ALEXANDRA BAC - LIS MARSEILLE

From the geometry of proofs to that of holes

Thomas Ehrhard's 60th Birthday
September 29-30th 2022

Overview

- **Topology and Morse theory**
- **Computational homology**
  - Algebraic approches
  - « Categorical » approach: reductions
  - Combinatorial approach: discrete Morse theory
- **Made in Marseille: Homological Discrete Vector Fields**

Topology and Morse theory

Topology and algebraic topology

Object

Algebraic object (group)

Topology

Properties

Topological space

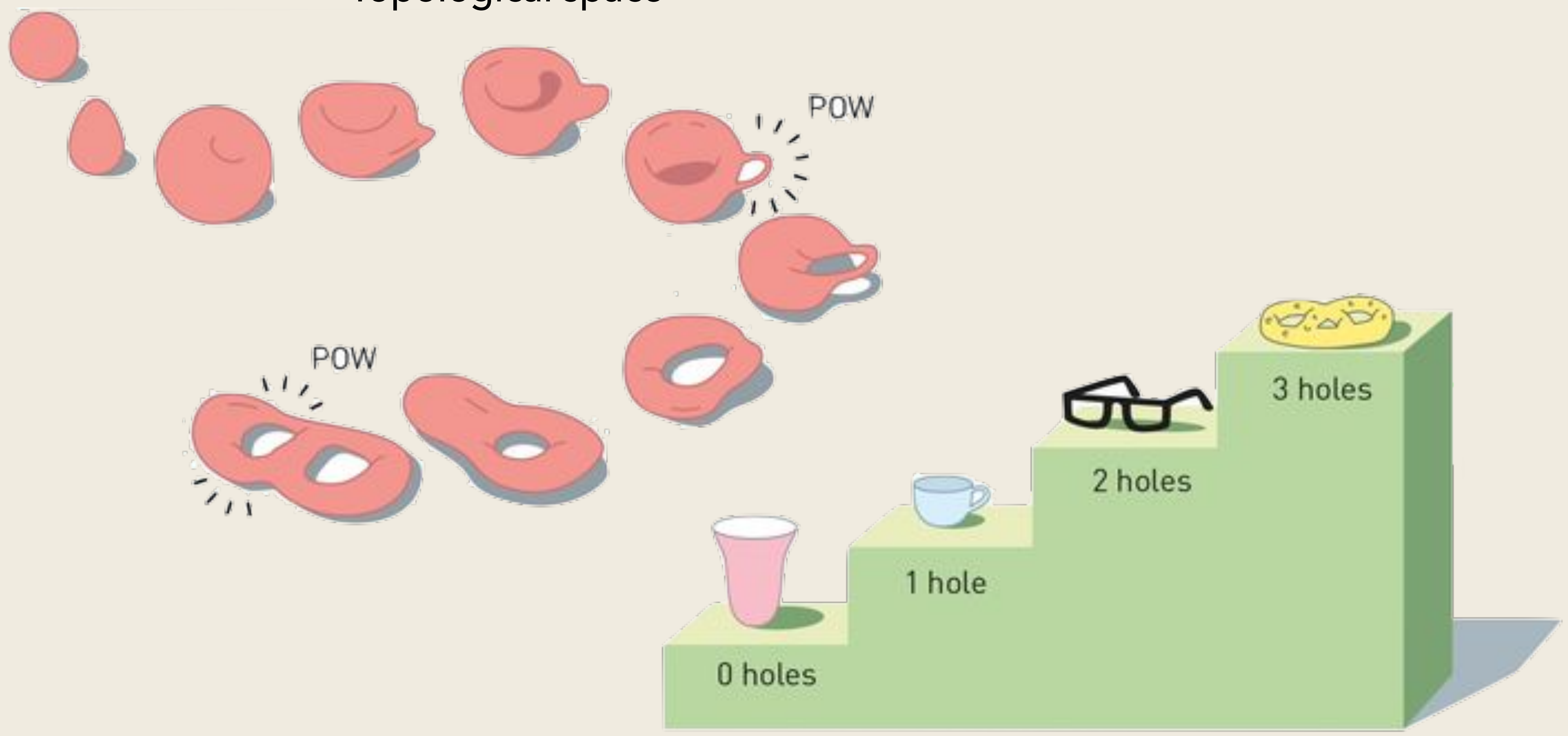
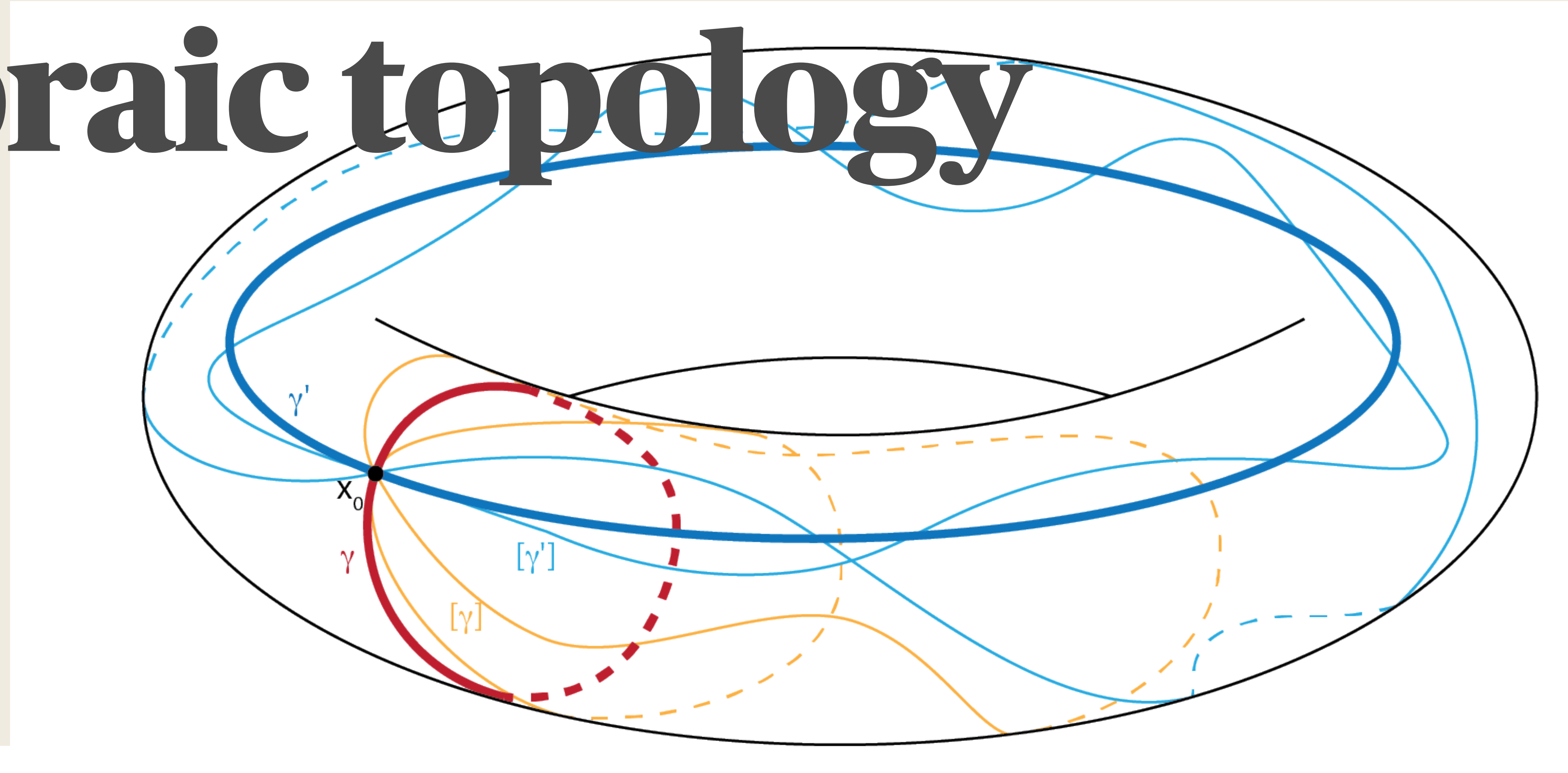


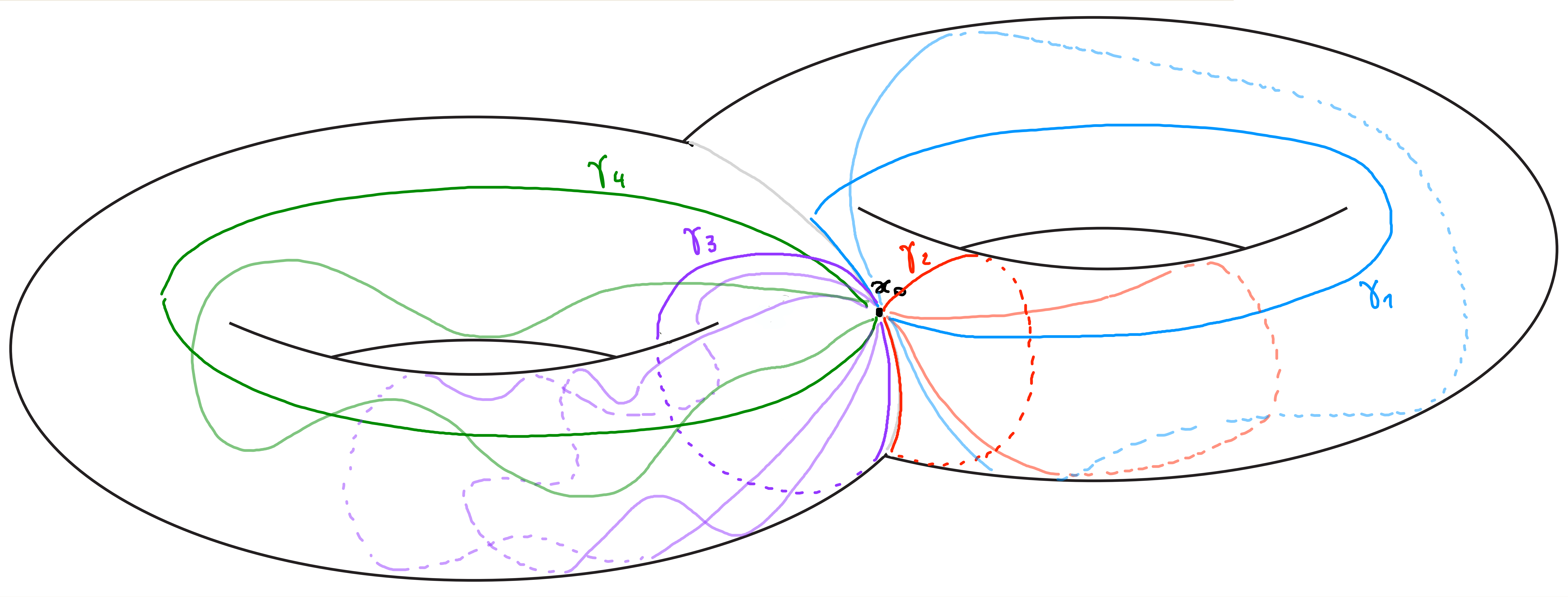
Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

Topology and algebraic topology



Homotopy

Fundamental group $(\Omega(X, x_0)/\sim_h, \circ)$



Topology and algebraic topology

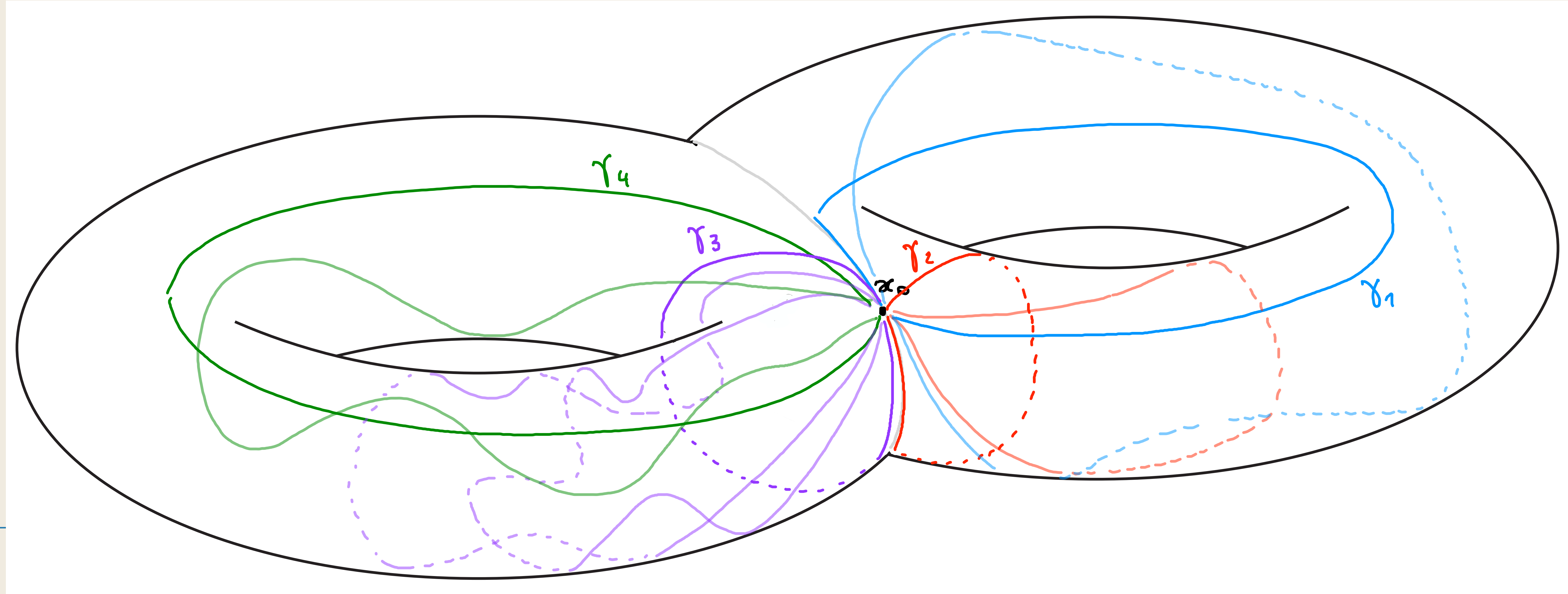
Homotopy

Fundamental group $(\Omega(X, x_0)/\sim_h, \circ)$

Computationally (th. Van Kampen)

$\rightarrow$ Group presentation

$$\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1} \rangle$$



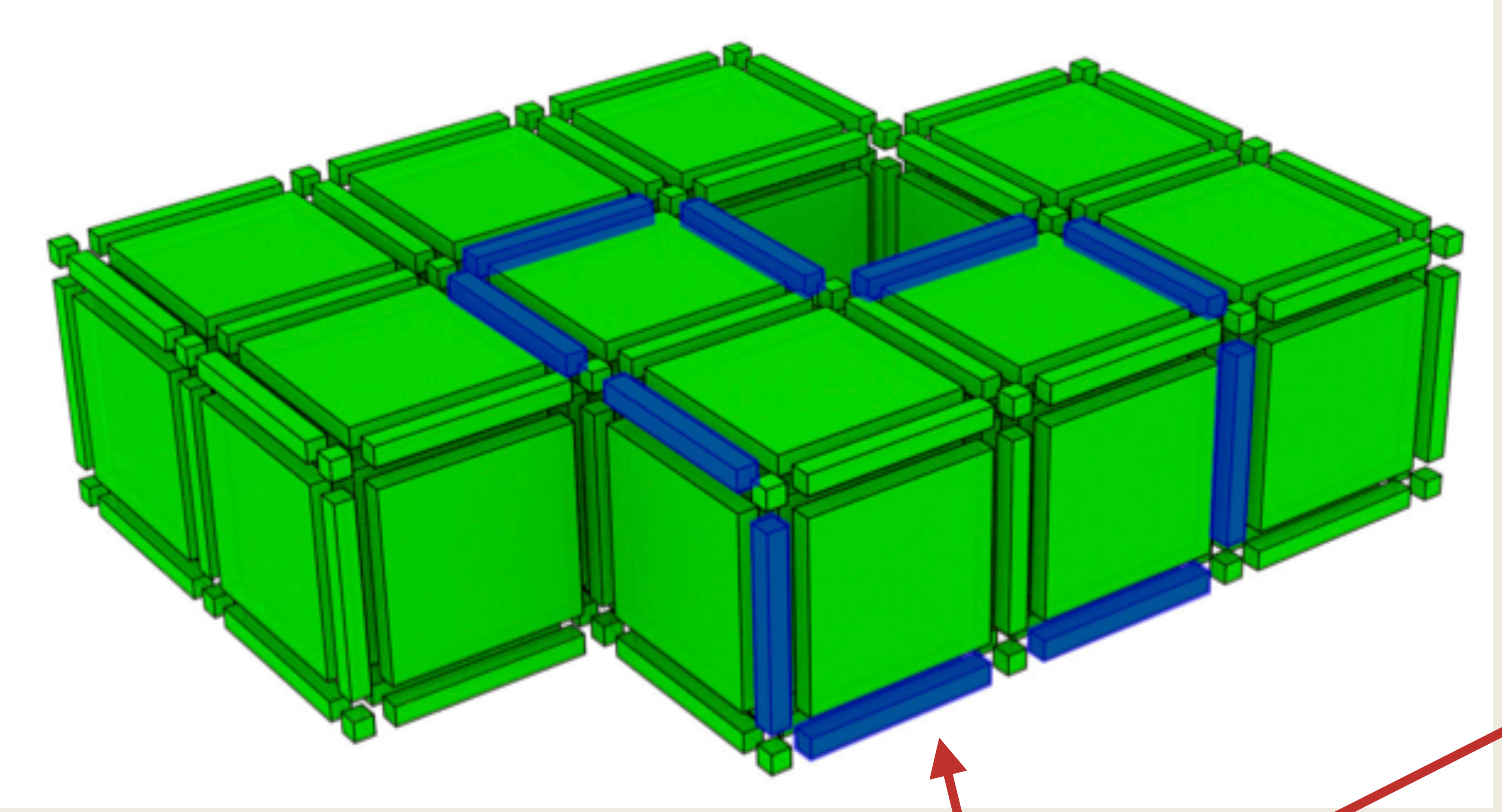
Th. Novikov-Boone

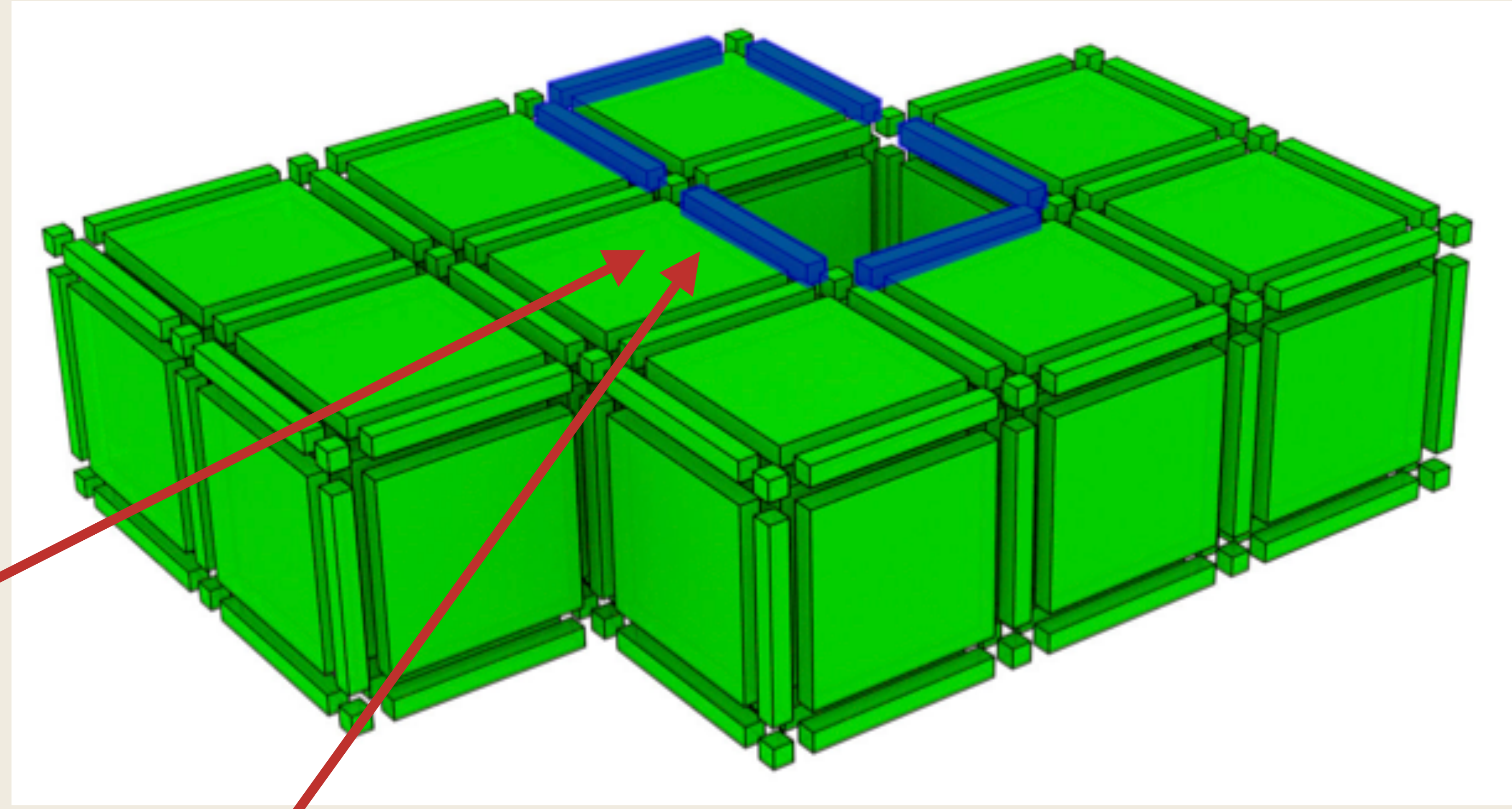
Deciding if a group is trivial given its presentation:

non decidable

(Simplicial) homology

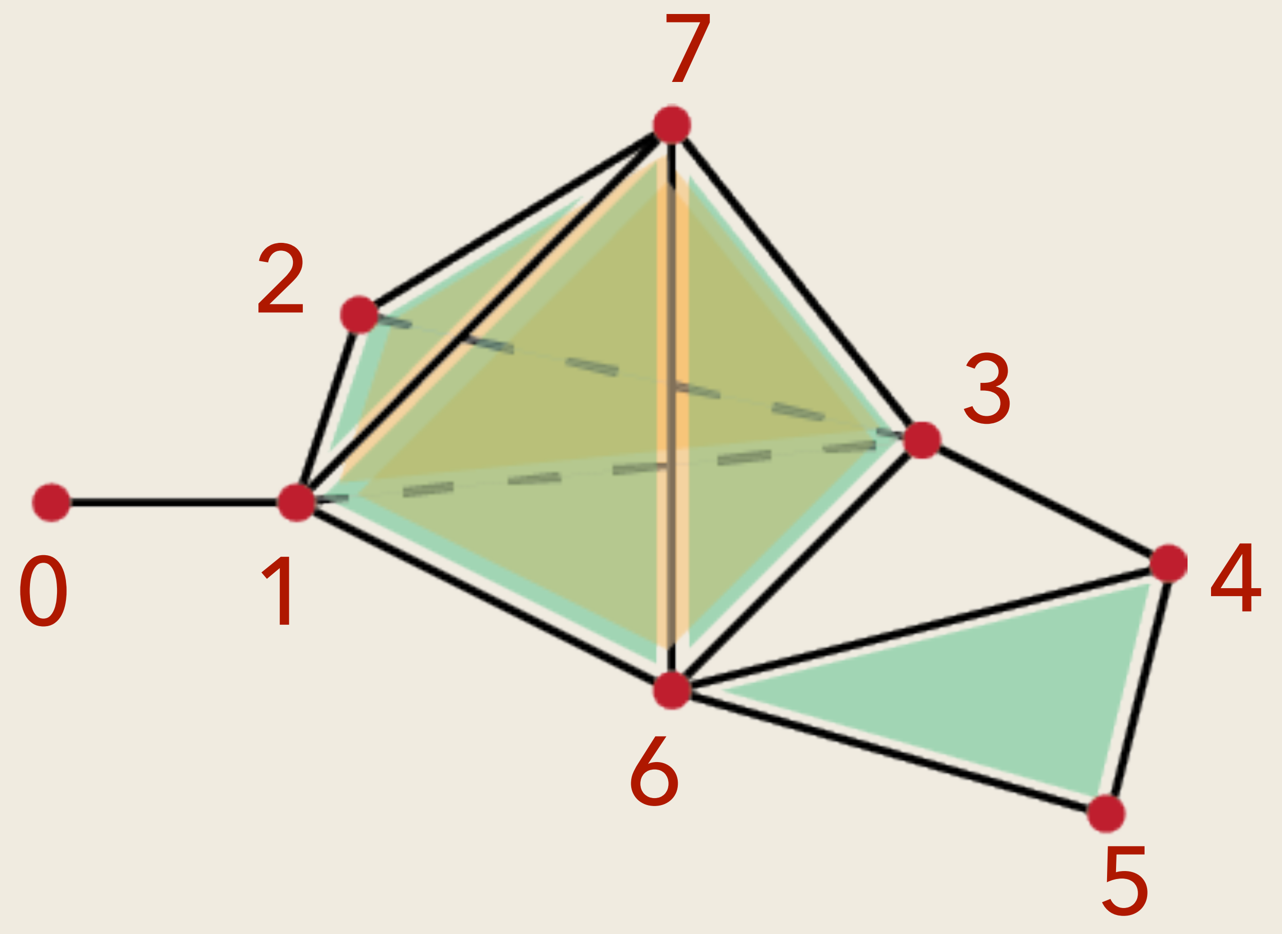
TOPOLOGY ← HOLES OF AN OBJECT





Hole: **cycle** which is **not the boundary of anything**

(Simplicial) homology



A a coefficient ring

$$\boxed{C_q} \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \cdots C_1 \xrightarrow{\partial_1} C_0$$

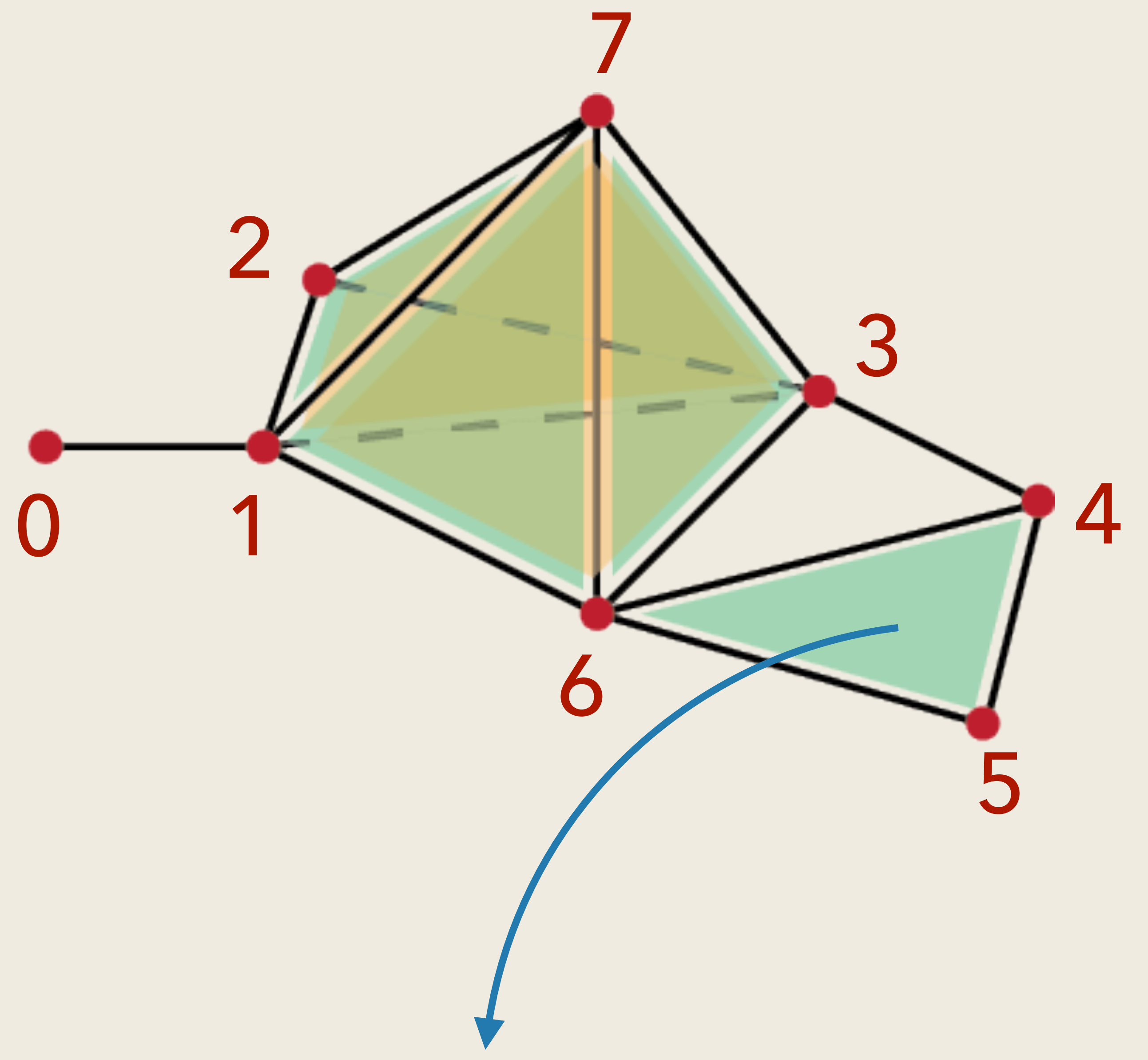
Chain complex

q**th chain group**: free group generated by q-cells with coefficients in A

Hole: **cycle** which is **not the boundary of anything**

(Simplicial) homology

A a coefficient ring



$$C_q \xrightarrow{\boxed{\partial_q}} C_{q-1} \xrightarrow{\partial_{q-1}} \cdots C_1 \xrightarrow{\partial_1} C_0$$

Chain complex

Boundary morphism

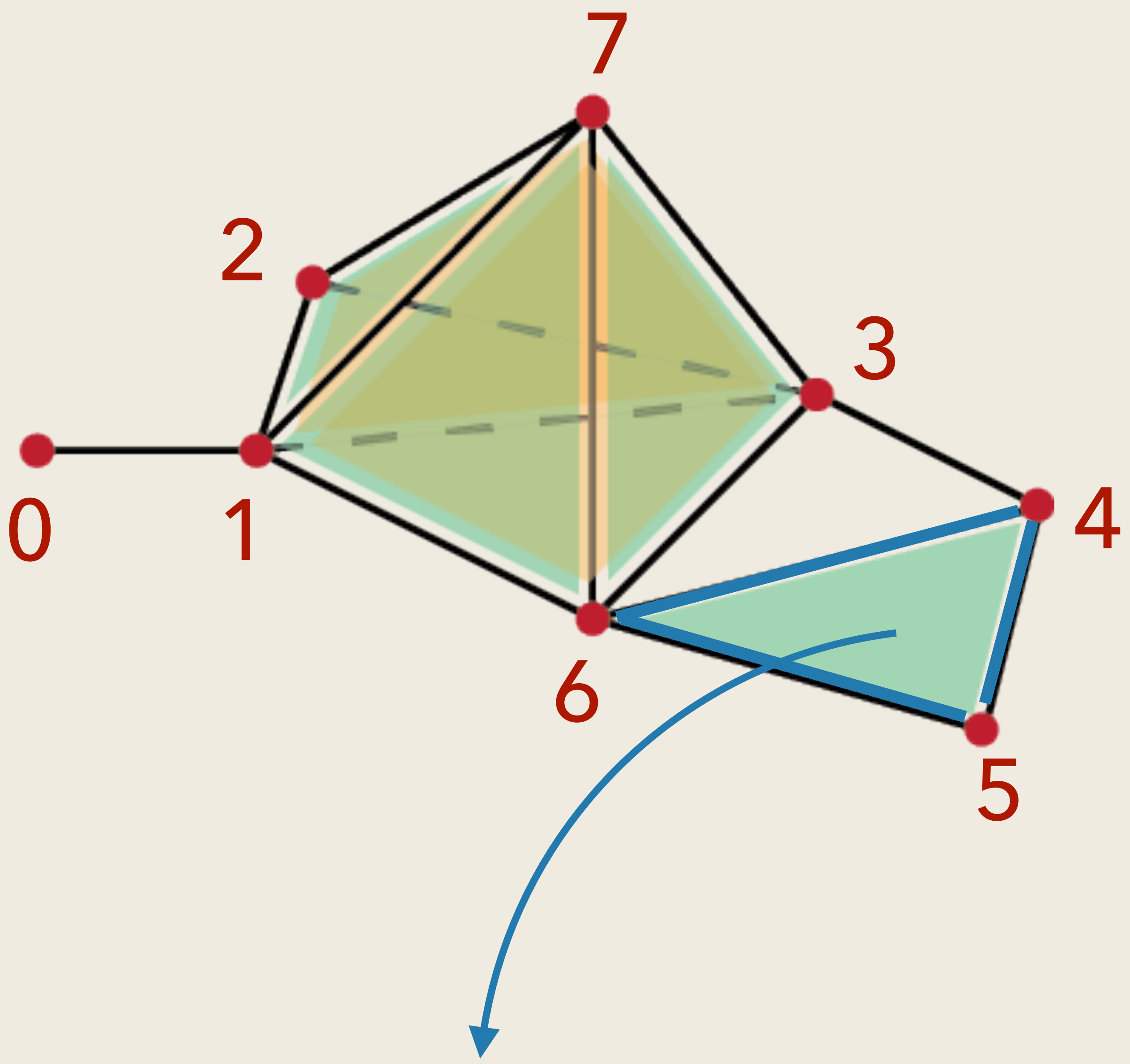
Satisfies: $\partial_q \partial_{q-1} = 0$

$\partial_2(f_{456}) = e_{56} - e_{46} + e_{45}$

$\partial_1(e_{45}) = v_5 - v_4 \ldots$

Hole: **cycle** which is **not the boundary of anything**

(Simplicial) homology



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A a coefficient ring

$$C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \cdots C_1 \xrightarrow{\partial_1} C_0$$

Chain complex

q-**cycles**: $\ker \partial_q$

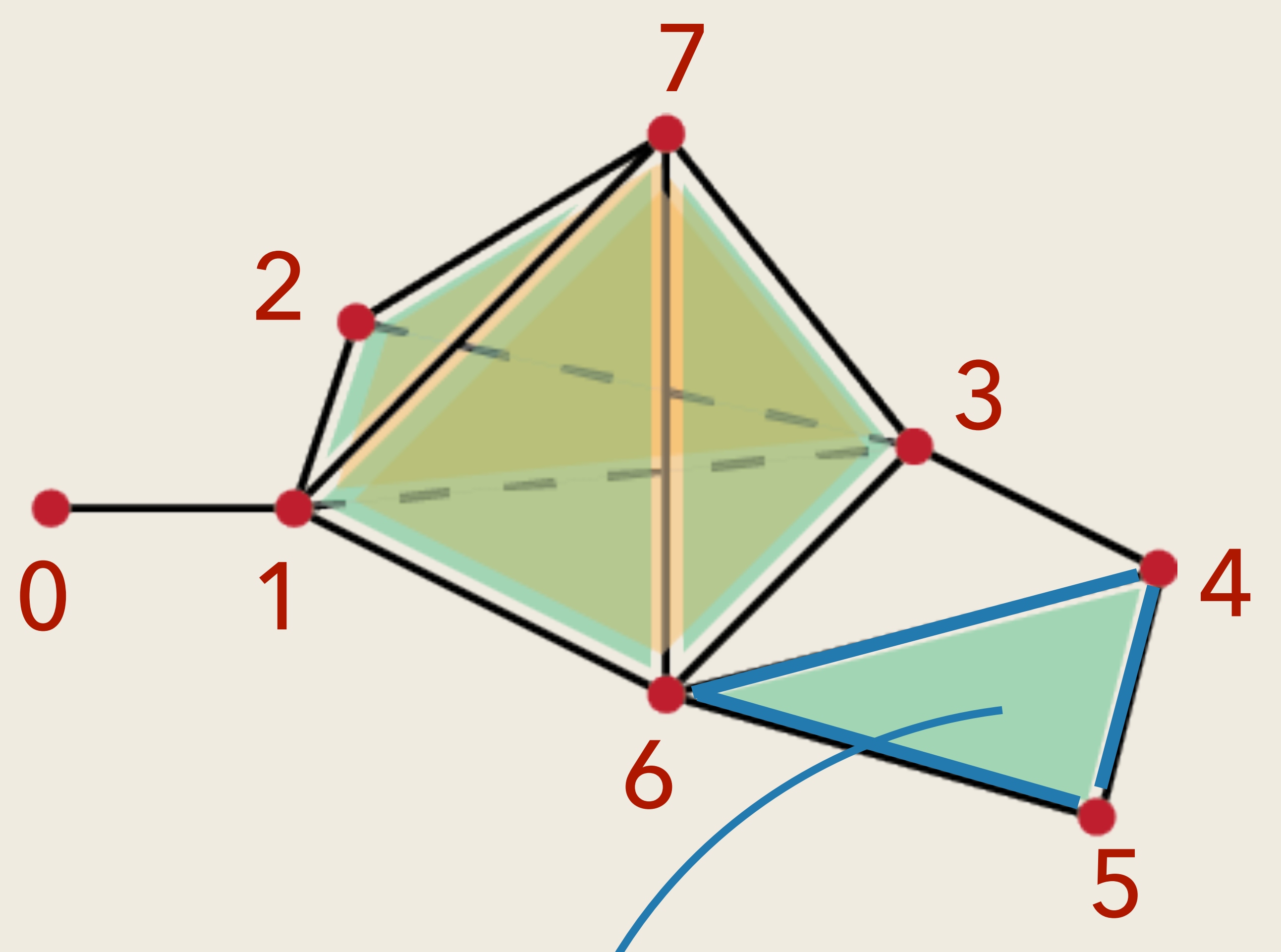
Boundaries: $\mathrm{Im}\, \partial_{q+1}$

$$\mathrm{Im}\, \partial_{q+1} \subseteq \ker \partial_q$$

Hole: **cycle** which is **not the boundary of anything**

(Simplicial) homology

A a coefficient ring



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Chain complex

$\partial_2(f_{456}) = e_{56} - e_{46} + e_{45}$

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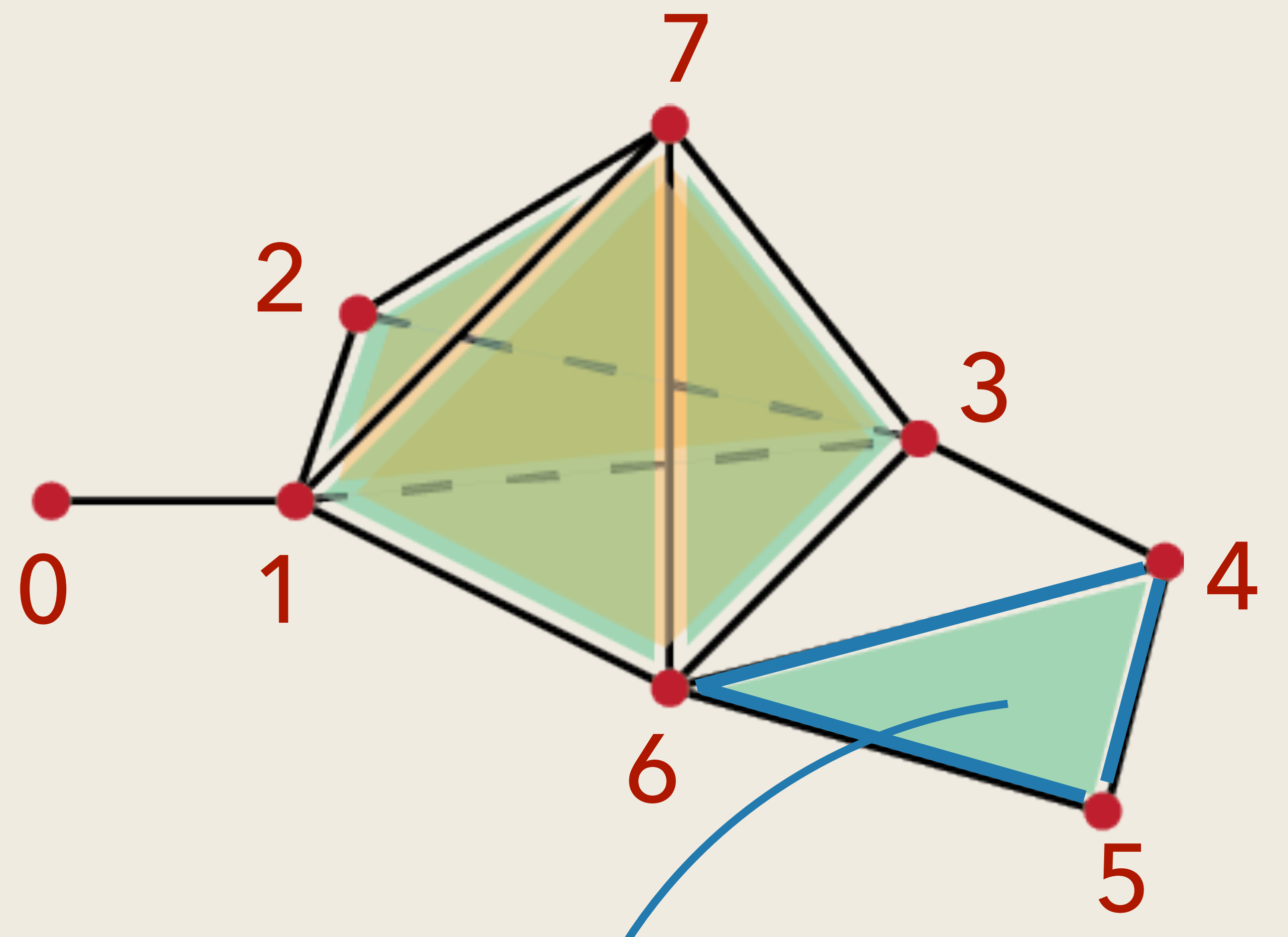
q**-th homology group**:

$$H_q(C) = \ker \partial_q / \mathrm{Im}\, \partial_{q+1}$$

Hole: **cycle** which is **not the boundary of anything**

(Simplicial) homology

A a coefficient ring



$$C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \cdots C_1 \xrightarrow{\partial_1} C_0$$

Chain complex

$$\partial_2(f_{456}) = e_{56} - e_{46} + e_{45}$$

$$\partial_1(e_{45}) = v_5 - v_4 \ldots$$

q**-th homology group**:

$$H_q(C) = \ker \partial_q / \operatorname{Im} \partial_{q+1}$$

$$H_q(C) \simeq \mathbb{Z}^{\boxed{\beta_q}} \times \mathbb{Z}/\lambda_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\lambda_n\mathbb{Z} \qquad \lambda_i \mid \lambda_{i+1}$$

q-th Betty number: number of holes of dimension q

Computational homology

Computational homology

Homology is computable in polynomial time (Munkres, 1984)

ALGEBRAIC APPROACHES

SMITH NORMAL FORM

Pseudo-diagonalisation of the boundary matrix

CATEGORICAL APPROACHES

EFFECTIVE HOMOLOGY

Computation of a reduction to a smaller complex

COMBINATORIAL APPROACHES

DISCRETE MORSE THEORY

Computation of graph on the complex

Upper bound on Betty numbers

Geometry

« Categorical » approches : reductions and effective homology

Reduction between two chain complexes(C, ∂) and (C', ∂'):

$\rho = (h, f, g)$

with $h_q : C_q \to C_{q+1}$, $f : C \to C'$, $g : C' \to C$

1. f, g chain morphisms
2. $fg = \mathrm{Id}_{C'}$
3. $gf = \mathrm{Id}_C + h\partial + \partial h$
4. $hh = 0, fh = 0, hg = 0$

$$
\begin{tikzcd}
{} \arrow[r] & C_{q+1} \arrow[r, shift left, "\partial_{q+1}"] \arrow[d, shift right, "f_{q+1}"'] & C_q \arrow[l, shift left, "h_q"] \arrow[r, shift left, "\partial_q"] \arrow[d, shift right, "f_q"'] & C_{q-1} \arrow[l, shift left, "h_{q-1}"] \arrow[r] \arrow[d, shift right, "f_{q-1}"'] & {} \\
{} \arrow[r] & C'_{q+1} \arrow[r, "\partial'_{q+1}"] \arrow[u, shift right, "g_{q+1}"'] & C'_q \arrow[r, "\partial'_q"] \arrow[u, shift right, "g_q"'] & C'_{q-1} \arrow[r] \arrow[u, shift right, "g_{q-1}"'] & {}
\end{tikzcd}
$$

(Sergeraert, 2002)

Then:

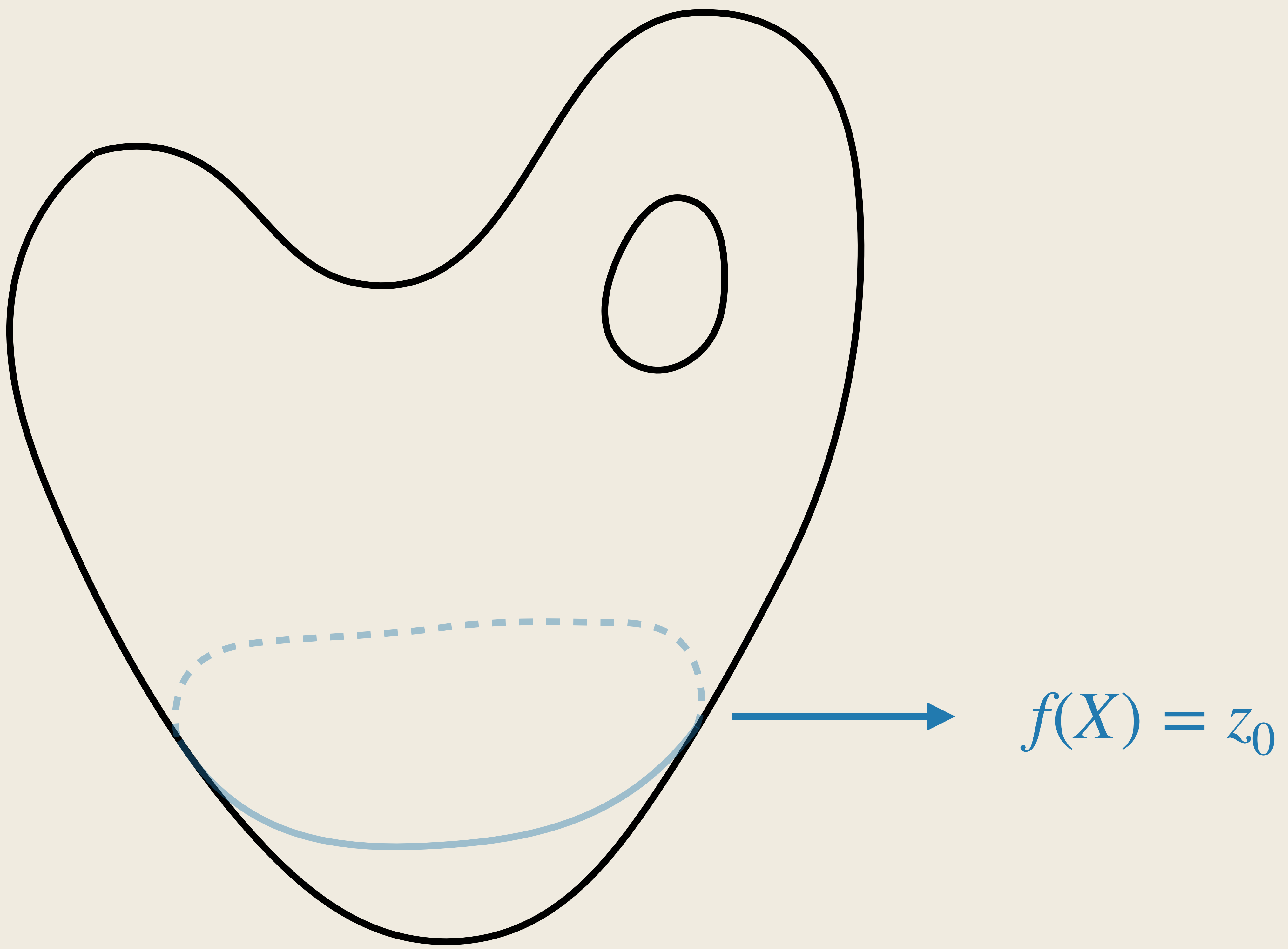
$\forall q \quad H_q(C) = H_q(C')$

Morse theory in 2 slides (and a half)

$\mathscr{M}$ a smooth closed manifold

$f : \mathscr{M} \to \mathbb{R}$ a smooth function

Ex: $f(X) = z$ height function



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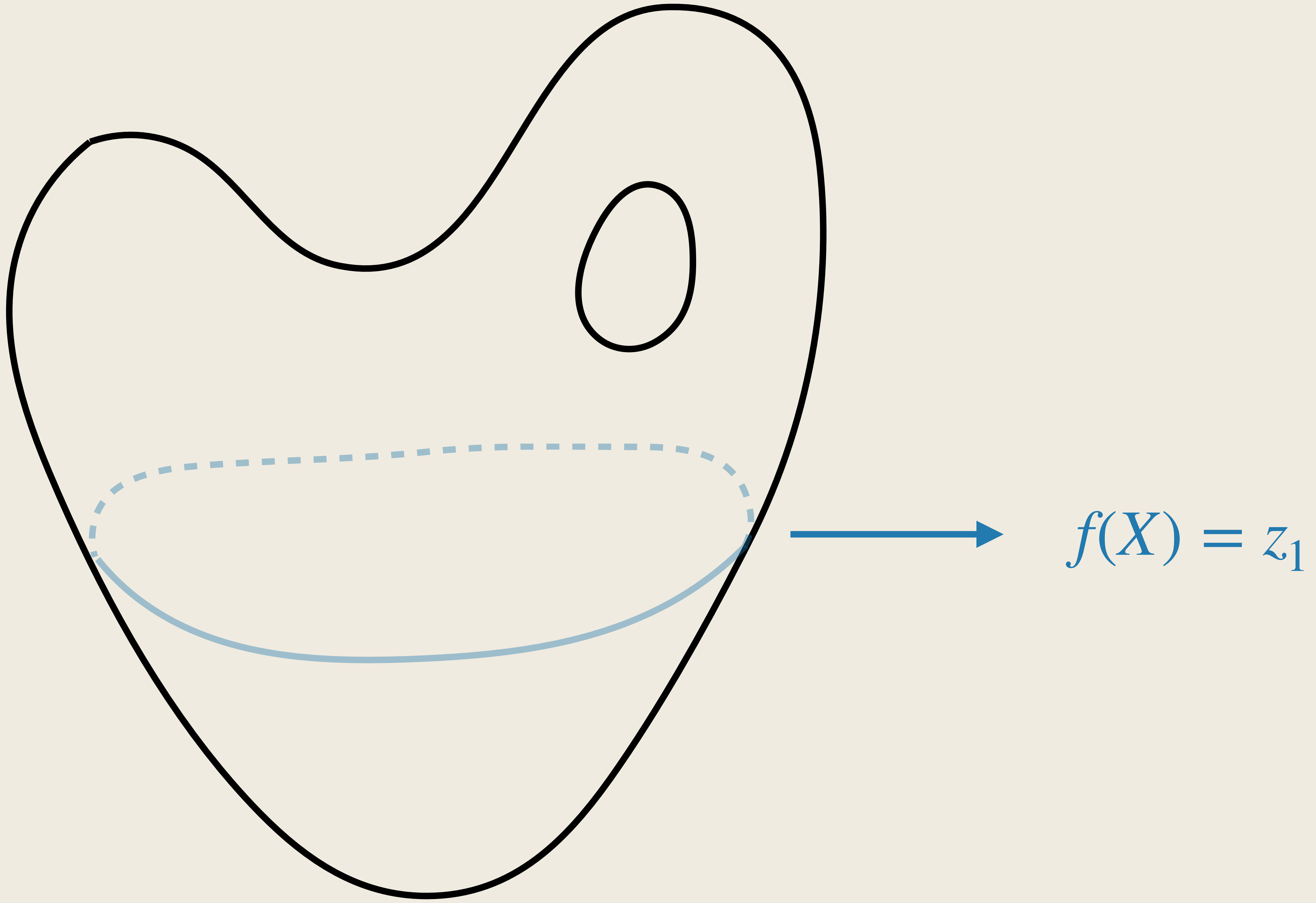

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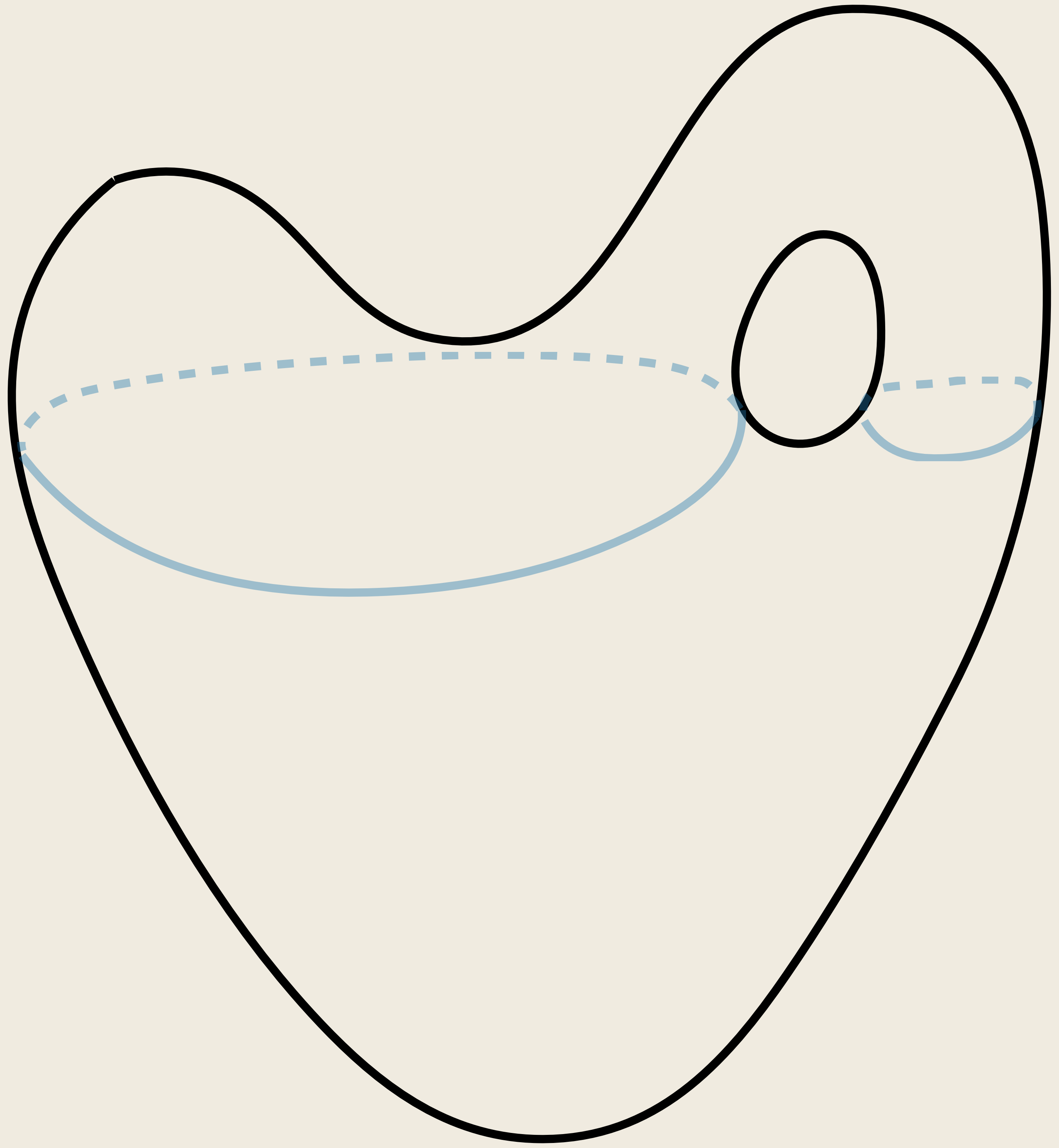

$f(X) = z_2$

Morse theory in 2 slides (and a half)

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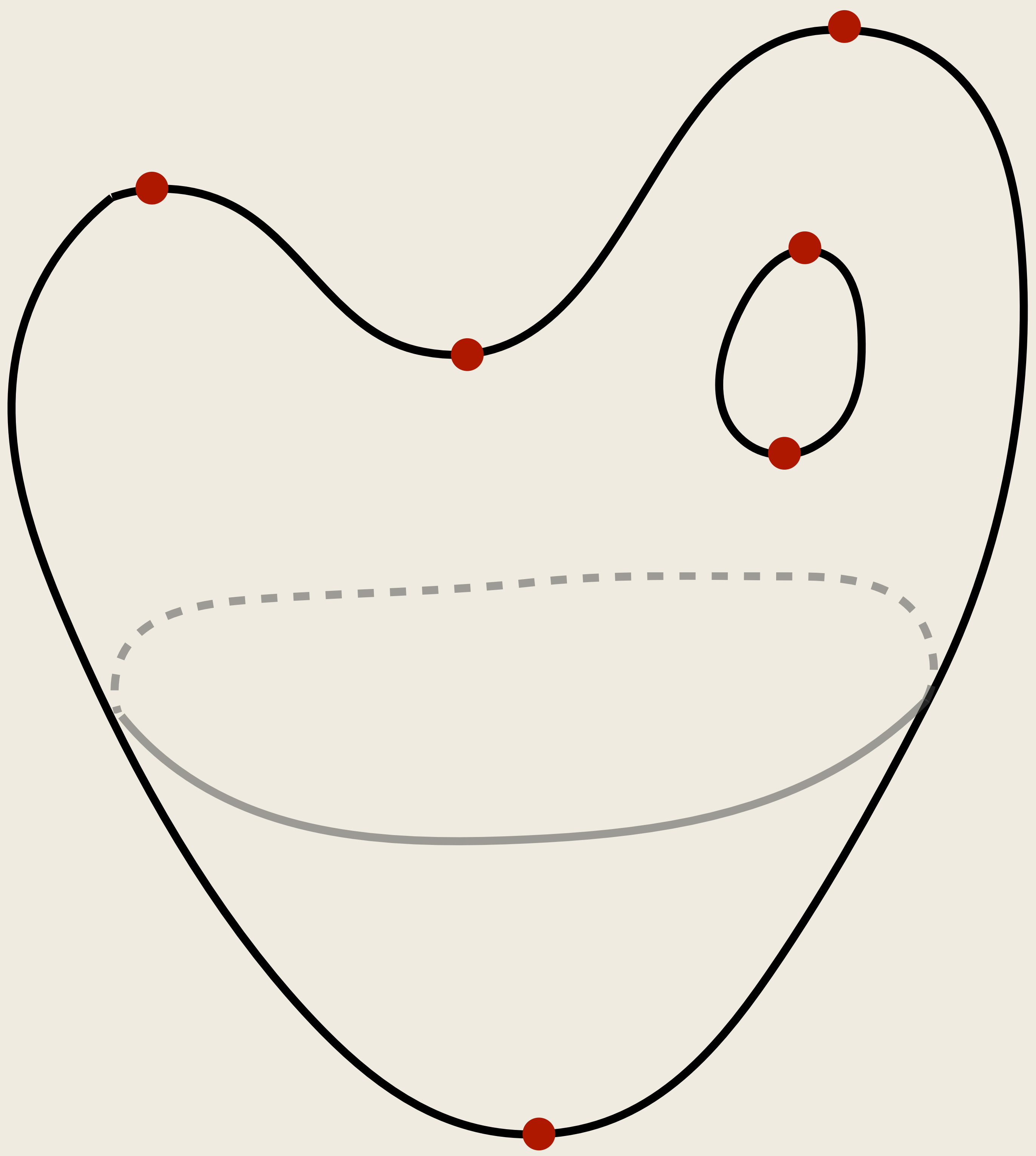


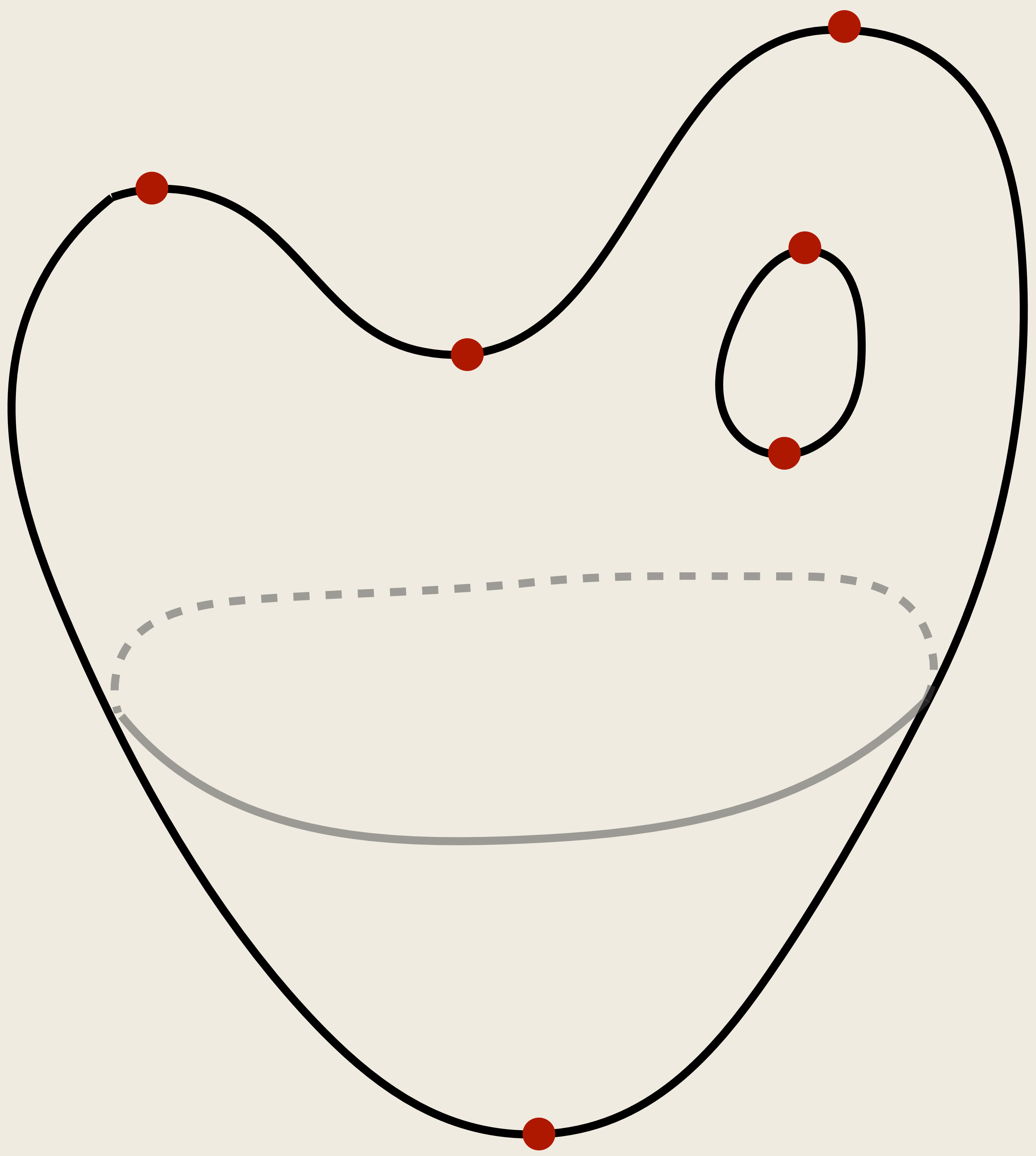
$f : \mathscr{M} \to \mathbb{R}$ a smooth function

Ex: $f(X) = z$ height function

X critical point if $\mathrm{d}f_X = 0$

Morse theory in 2 slides (and a half)

$\mathscr{M}$ a smooth closed manifold



$f : \mathscr{M} \to \mathbb{R}$ a smooth function

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X critical point if $\mathrm{d}f_X = 0$

X non degenerate critical point if $\mathrm{Hess}(f)_X$ non singular

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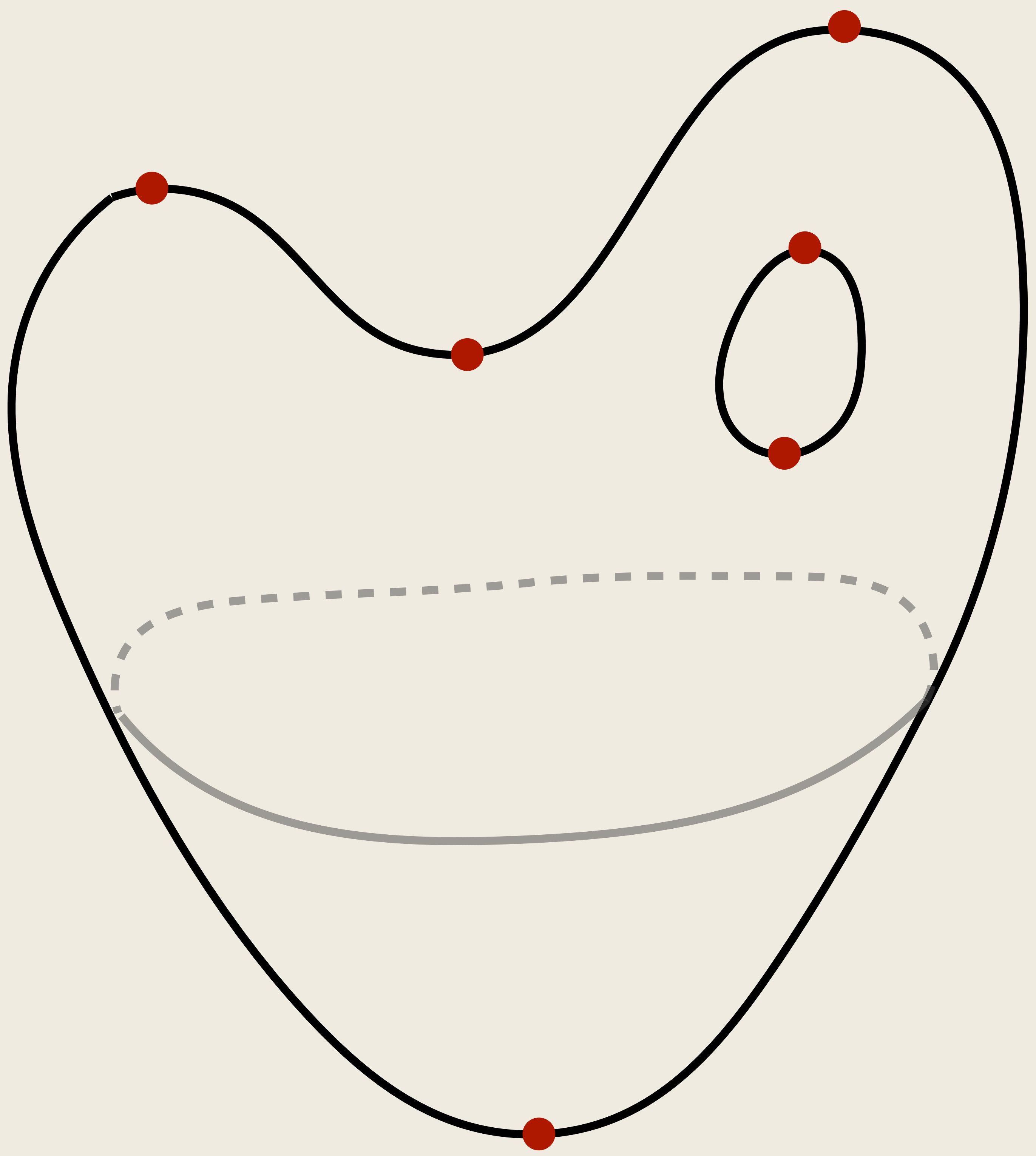

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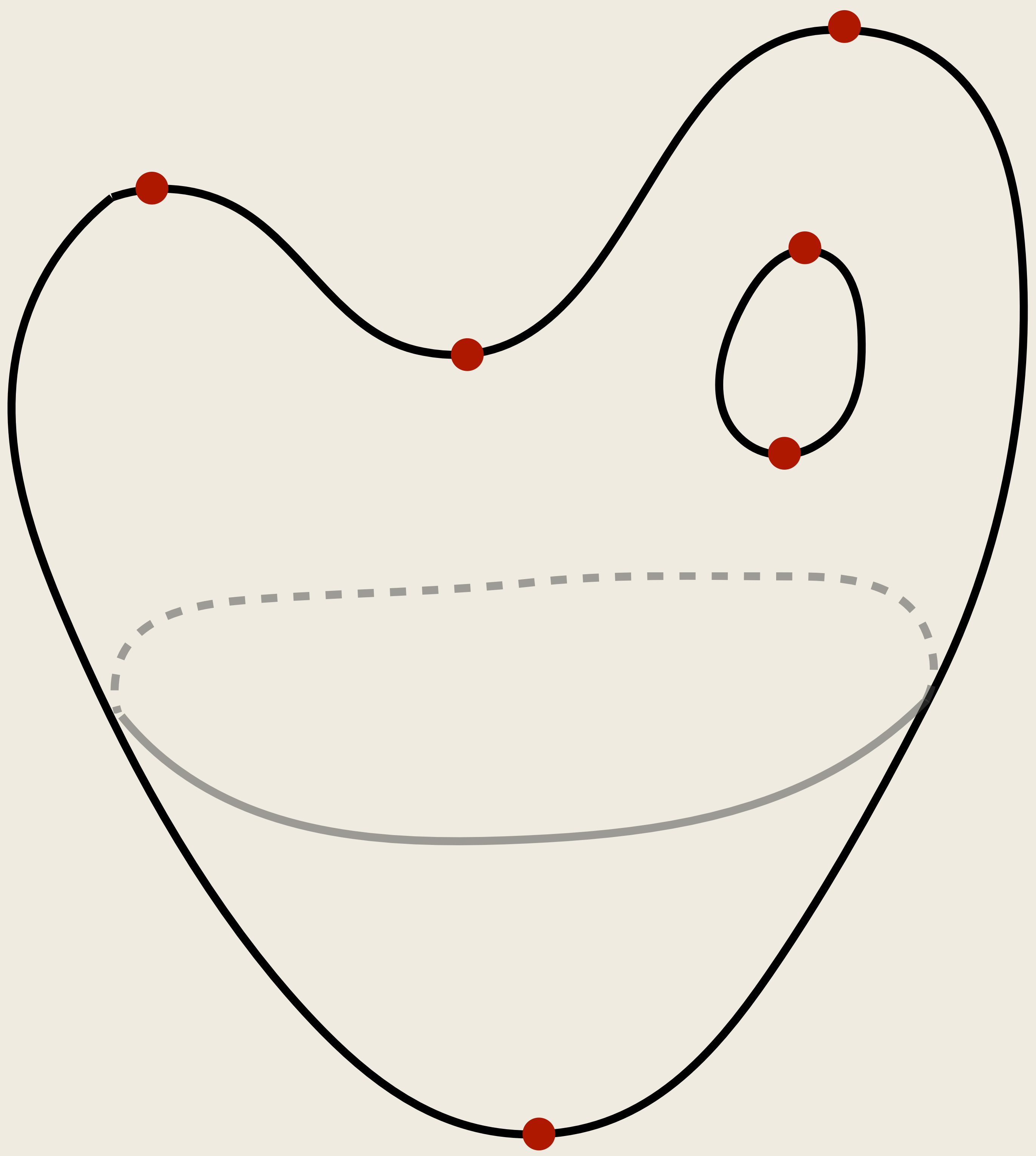
X non degenerate critical point if $\mathrm{Hess}(f)_X$ non singular

$f : \mathscr{M} \to \mathbb{R}$ is a Morse function on $\mathscr{M}$ if all critical points are non degenerate

Morse theory in 2 slides (and a half)

$\mathscr{M}$ a smooth closed manifold



$f : \mathscr{M} \to \mathbb{R}$ a Morse function

Ex: $f(X) = z$ height function

Morse lemma

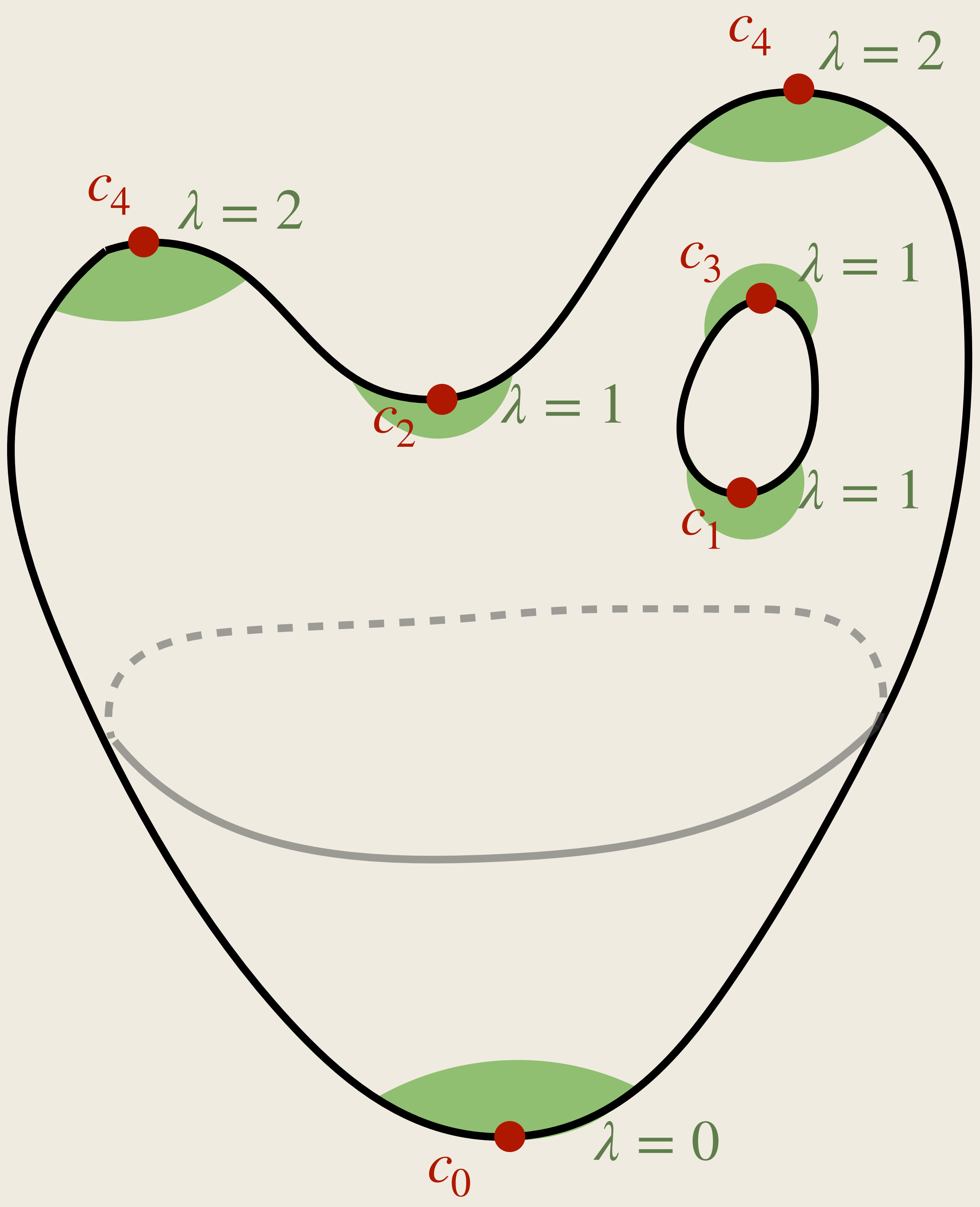
If X is a (non-degenerate) critical point, there exist a local chart $(x_1, \ldots, x_n)$ in a neighbourhood U of X s.t:

$$f(x_1, \ldots, x_n) = f(X) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2$$

Then λ index of f at X

Morse theory in 2 slides (and a half)

$\mathscr{M}$ a smooth closed manifold



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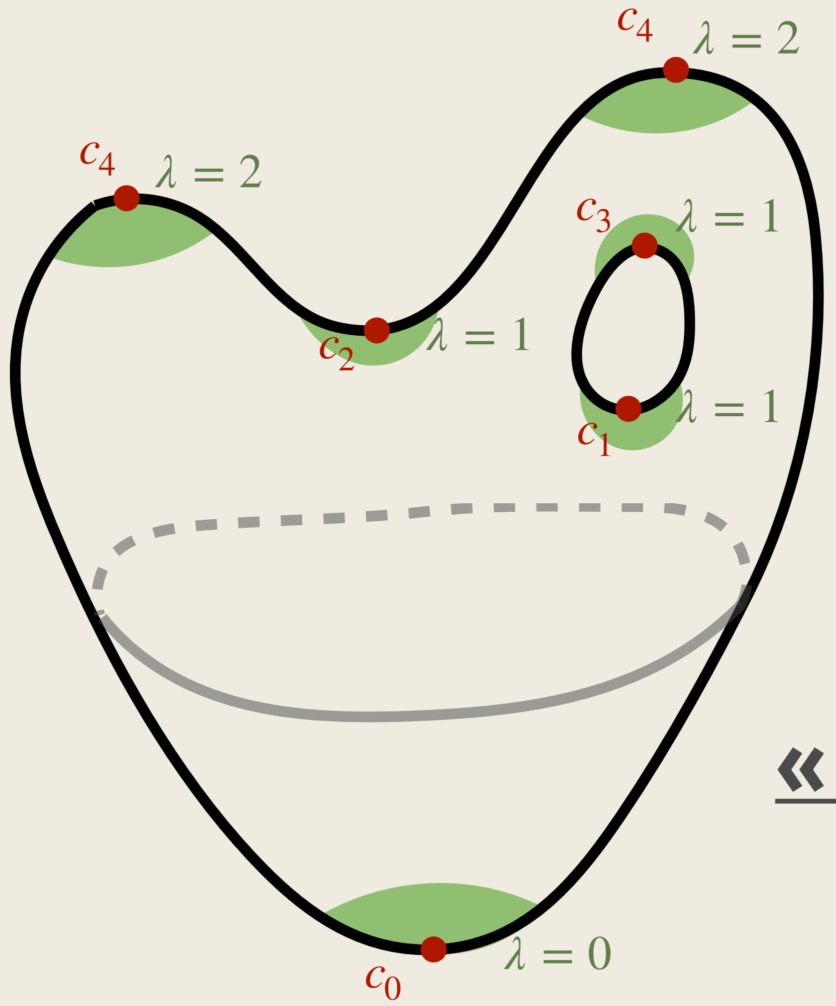


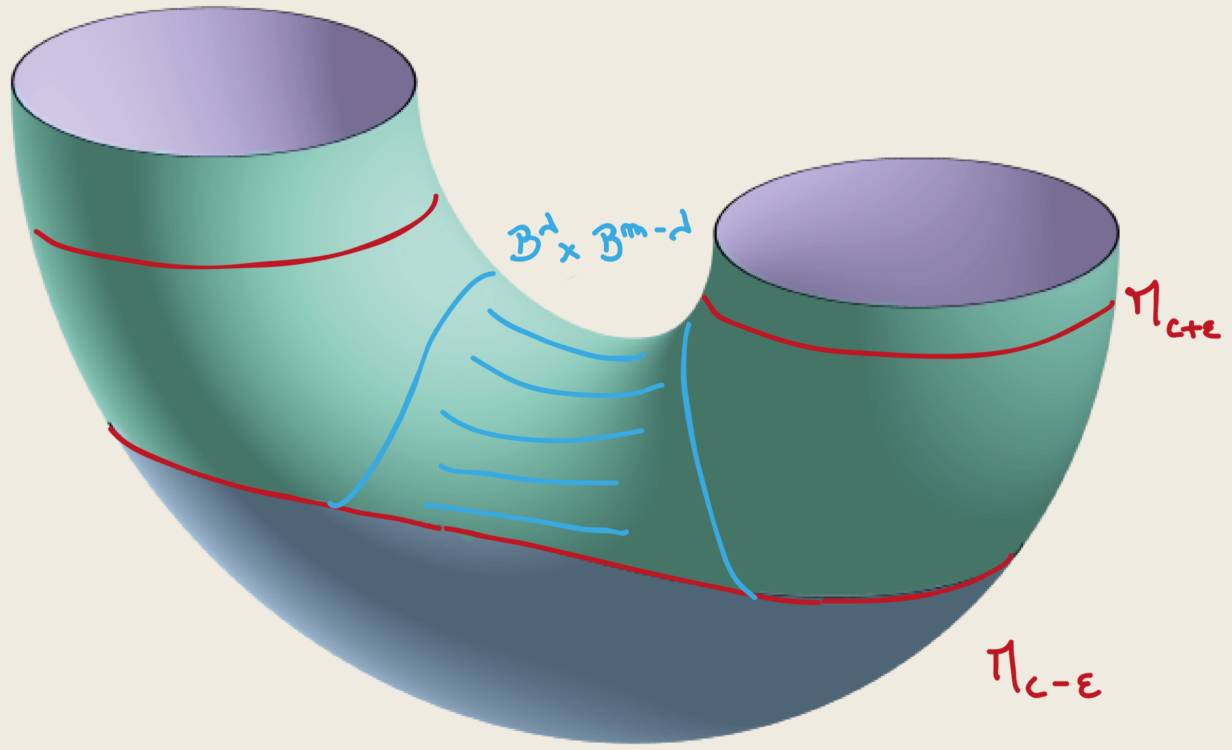
Morse theory in 2 slides (and a half)

« Corollaries »

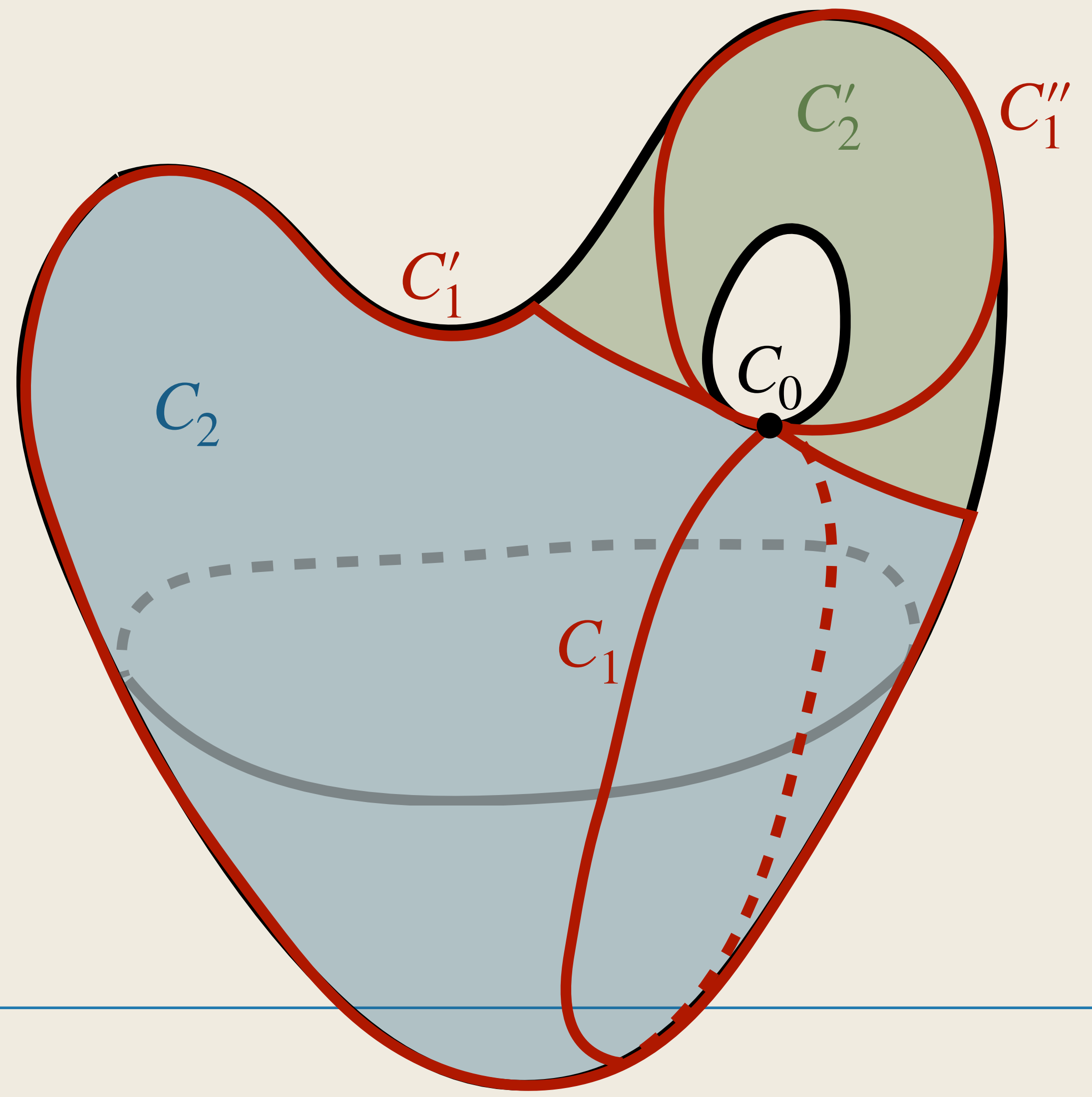
Surgery / decomposition of the manifold

Single critical value $c \in [c-\epsilon, c+\epsilon]$ of index λ

$$M_{c+\epsilon} \overset{\text{diffeo}}{\simeq} M_{c-\epsilon} \cup_g B^\lambda \times B^{m-\lambda}$$



CW-complex decomposition



Morse theory in 2 slides (and a half)

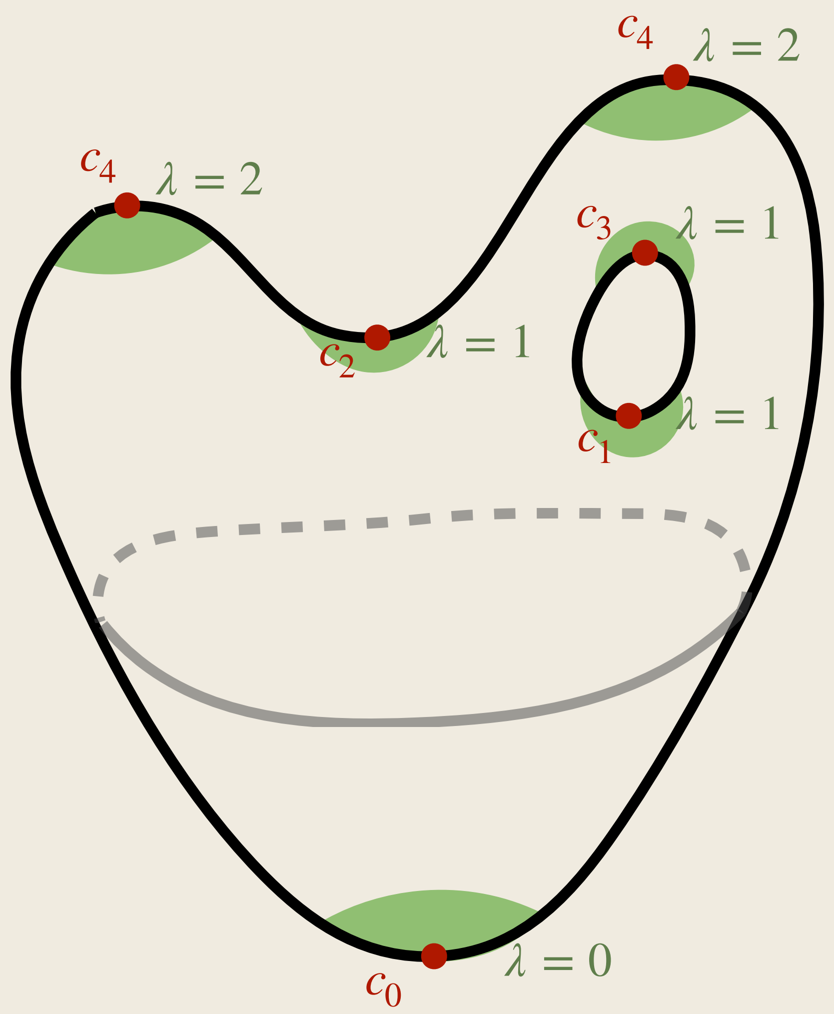
κ_q number of critical points of index q

Morse theorem
$\mathscr{M}$ closed m dimensional manifold

Euler-Poincaré invariant:

$$\chi(\mathscr{M}) = \sum_{q=0}^{m} (-1)^q \kappa_q$$

Where: $\chi(\mathscr{M}) = \sum_{q=0}^{m} (-1)^q \beta_q$



Weak Morse inequality

$$\forall d \quad \kappa_d \geqslant \beta_d$$

f is a perfect Morse function if

$$\forall q \quad \kappa_q = \beta_q$$

$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$

$\kappa_0 = 1$
$\kappa_1 = 3$
$\kappa_2 = 2$

$$\chi = 1 - 2 + 1 = 0 = 1 - 3 + 2$$

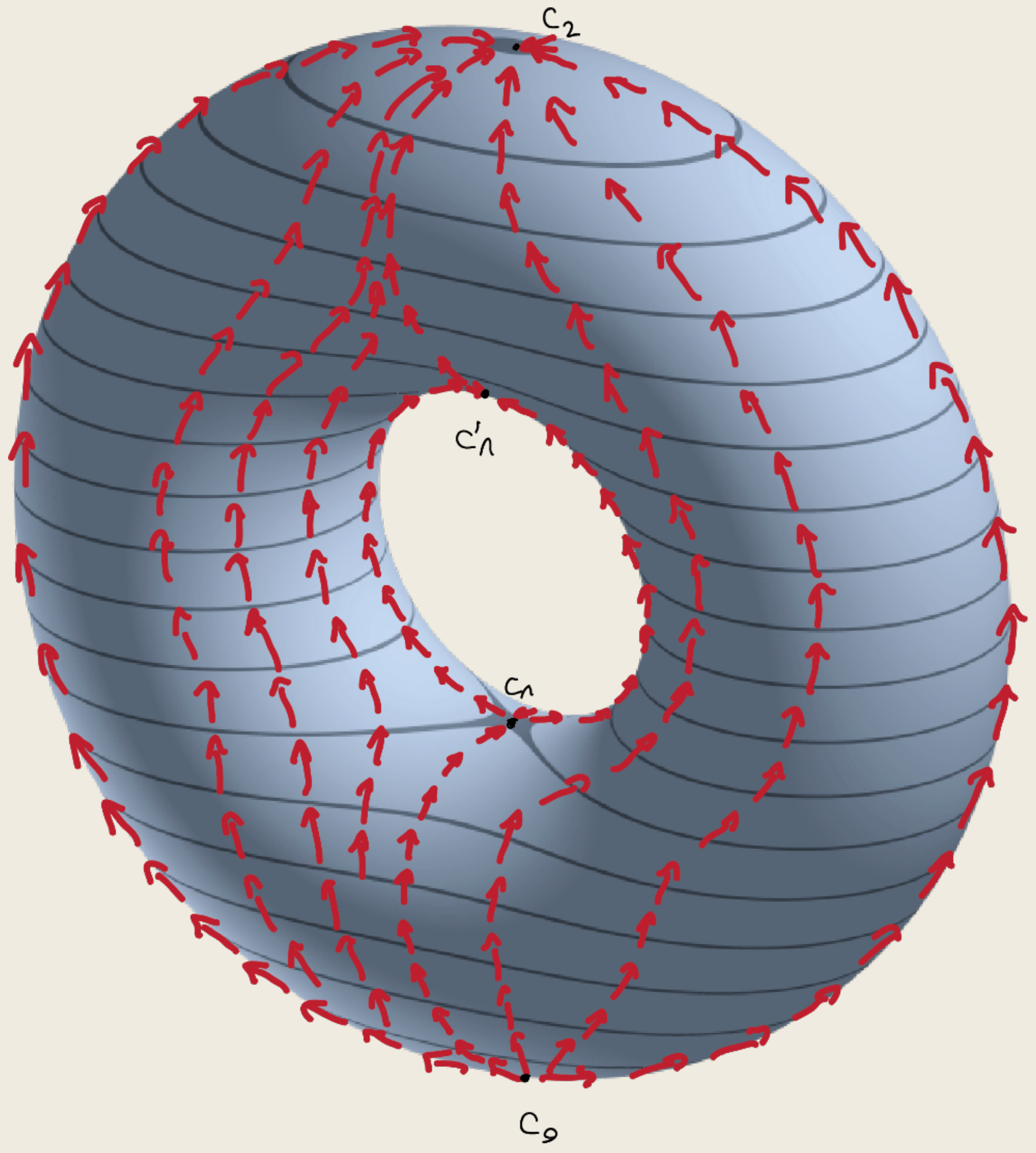
Morse theory in 2 slides (and a half)

Flow of the gradient vector field (**Morse-Smale**)

$\gamma_X : \mathbb{R} \to \mathscr{M}$ flow line trough X

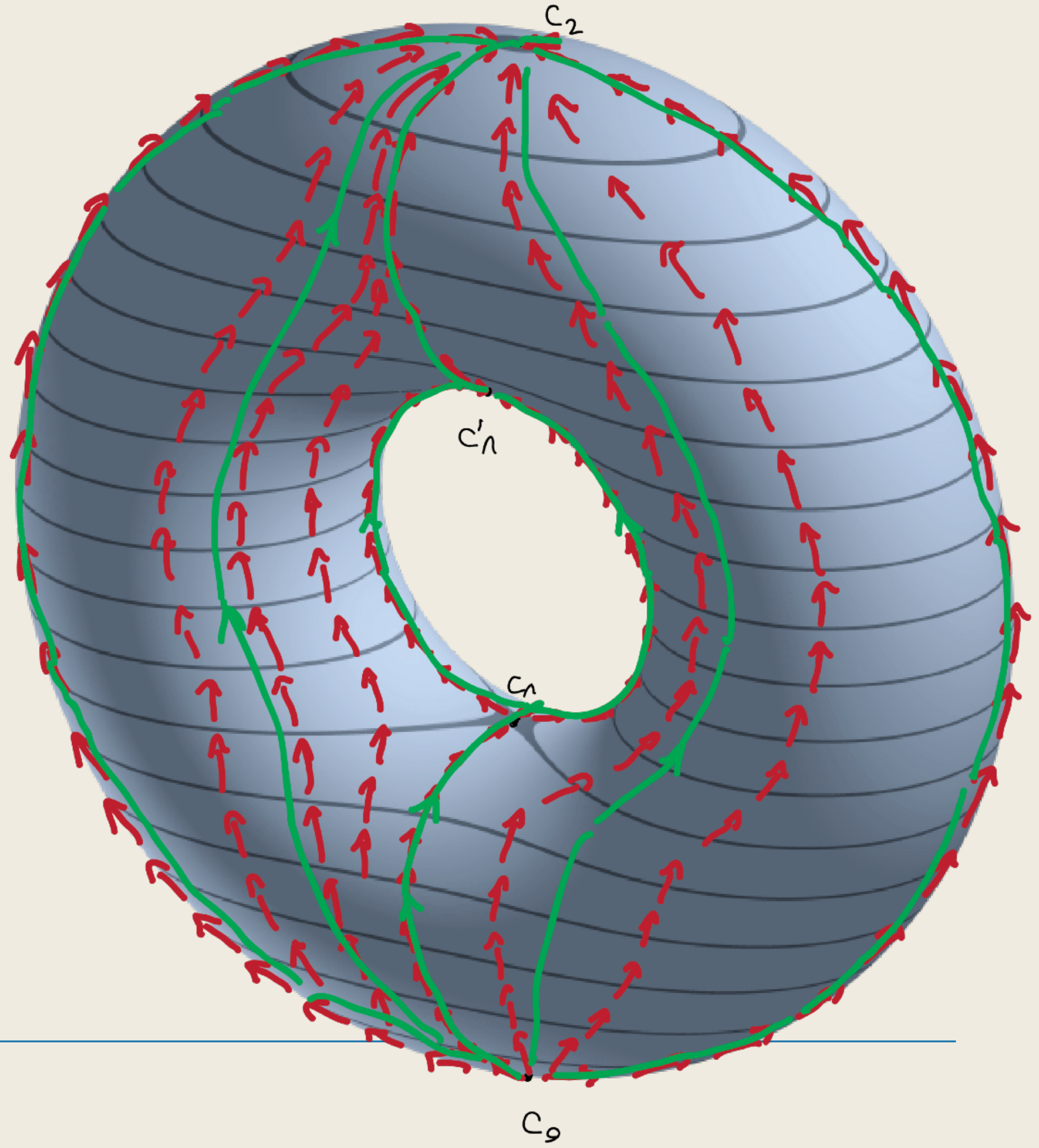
$$\frac{\mathrm{d}\gamma_X}{\mathrm{d}t} = \nabla_X f$$

and. $\gamma_X(0) = X$



∇f

Flow γ

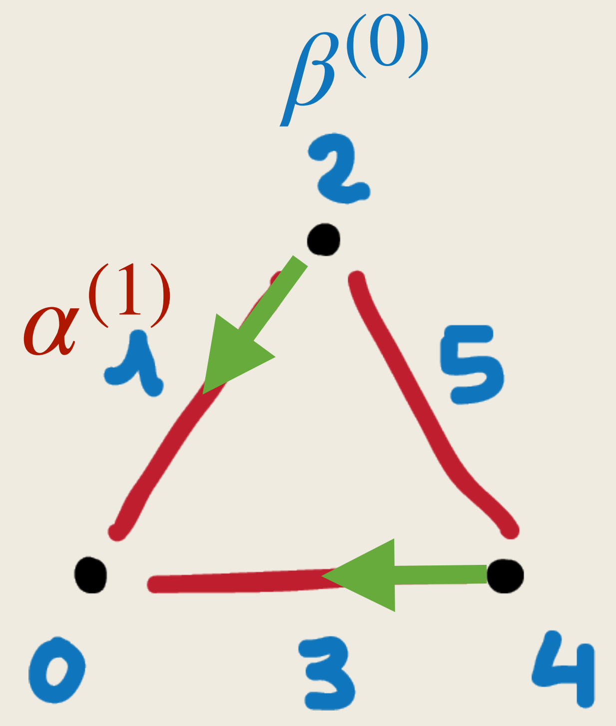


Discrete Morse theory
(R. Foreman, 1998)

Define **discrete Morse functions** over a simplicial complex K

- Better defined from their gradient vector field (V)
- By combinatorial properties

V matches
$\alpha^{(q)} \rightarrow \beta^{(q+1)}$



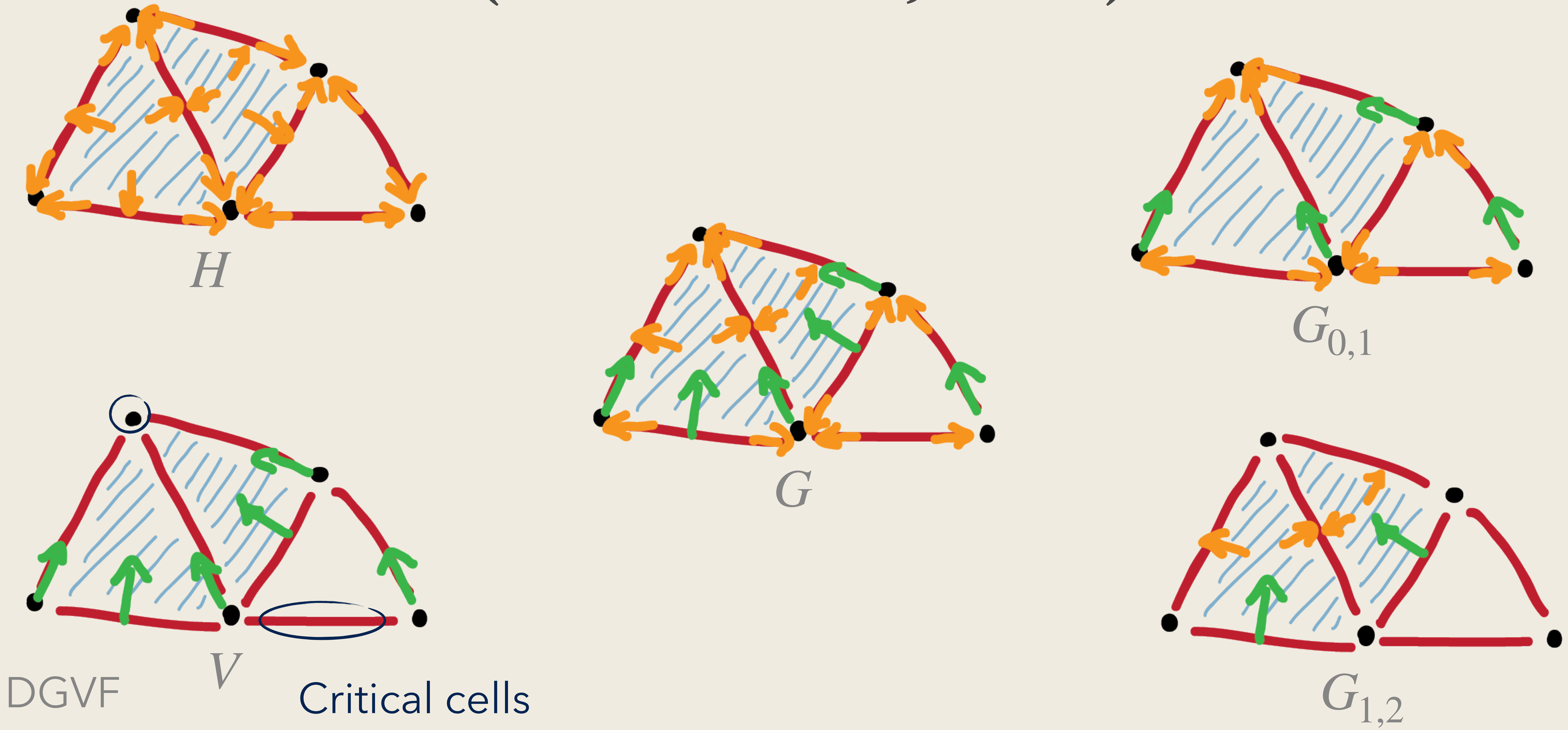
V **is a discrete gradient vector field** (DGVF) if:

→ V pairing on H (Hasse diagram of K - boundary graph)

→ Let G, obtained from H by inverting the edges of V

→ G is a Morse graph if all subgraphs $G_{q,q+1} = G|_{\dim q \cup \dim q+1}$ are acyclic (V-paths)

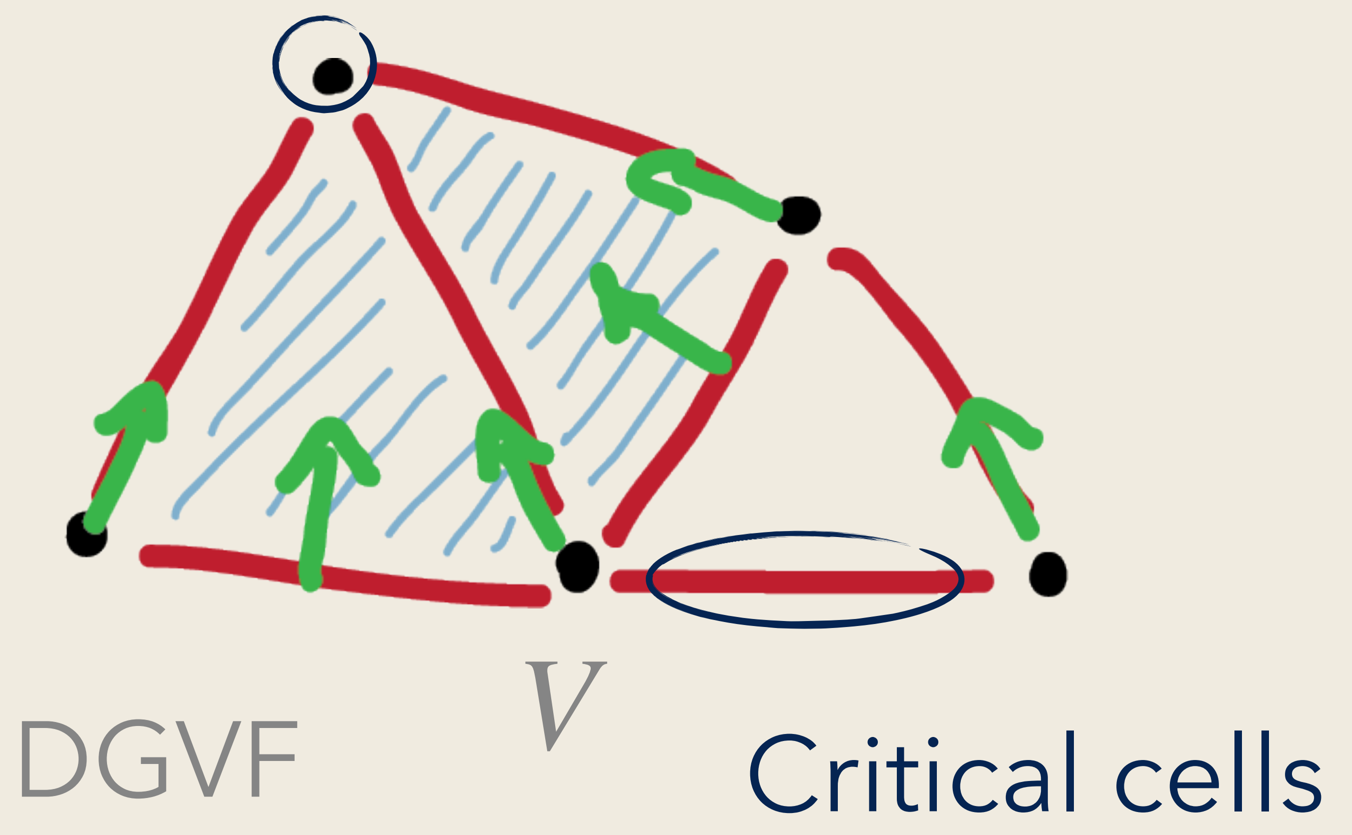
Discrete Morse theory
(R. Foreman, 1998)



Discrete Morse theory (R. Foreman, 1998)

V is **optimal** if κ_d minimal

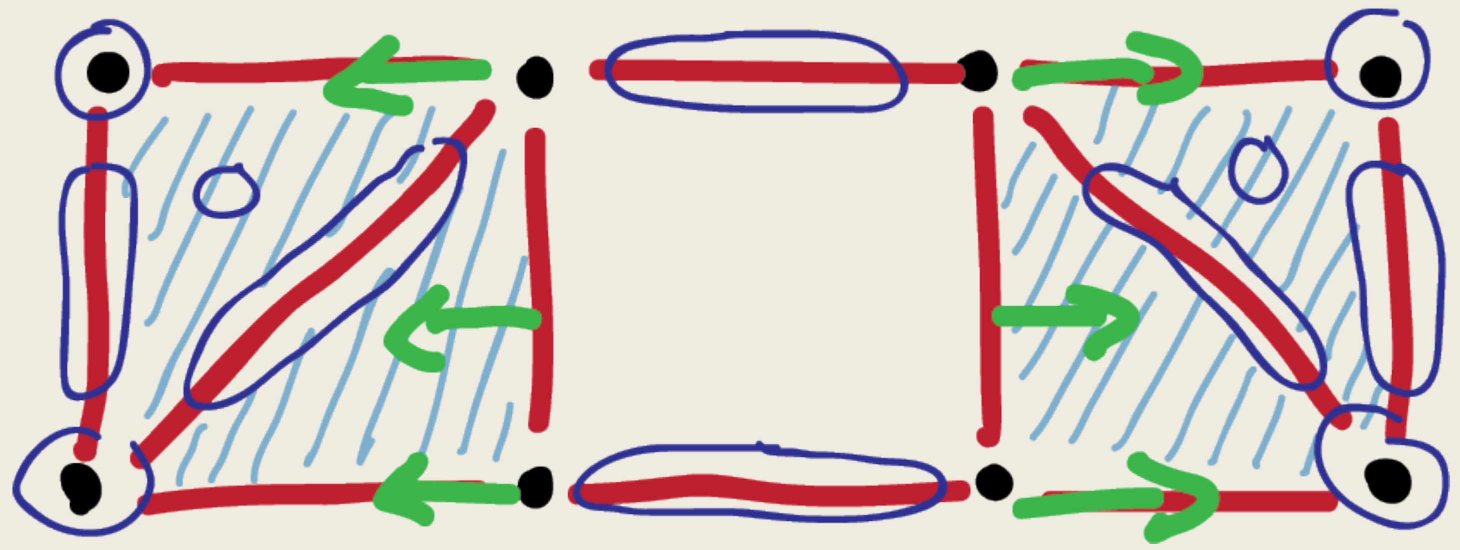
V is **perfect** if $\kappa_d = \beta_d \quad \forall d$



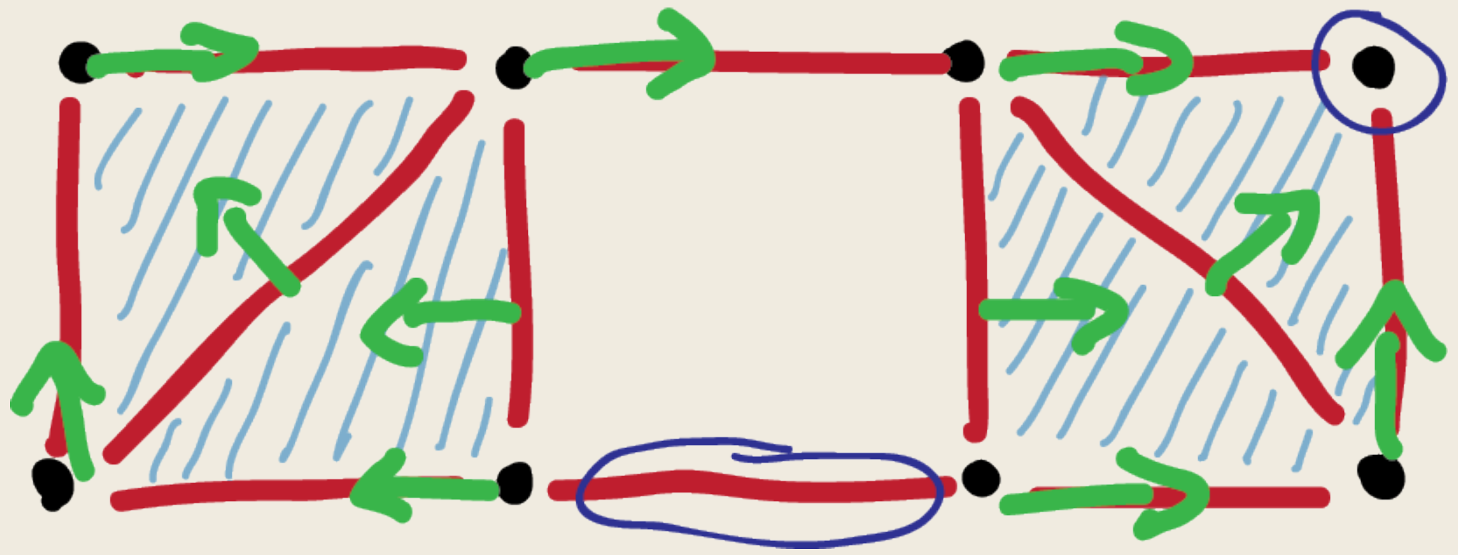
Discrete Morse lemma

$$\forall d \qquad \beta_d \leqslant \kappa_d$$

Discrete Morse theory
(R. Foreman, 1998)



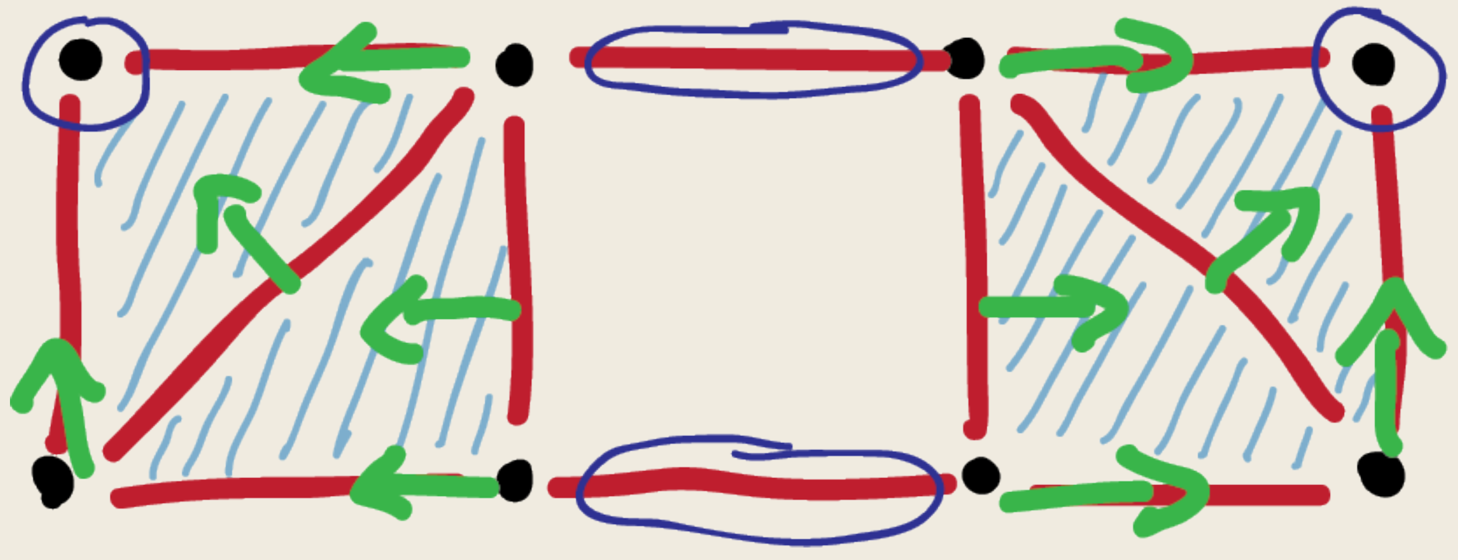
$\kappa_0 = 2 \qquad \beta_0 = 1$
$\kappa_1 = 2 \qquad \beta_1 = 1$



Perfect DGVF

IS A PERFECT DGVF COMPUTABLE ?

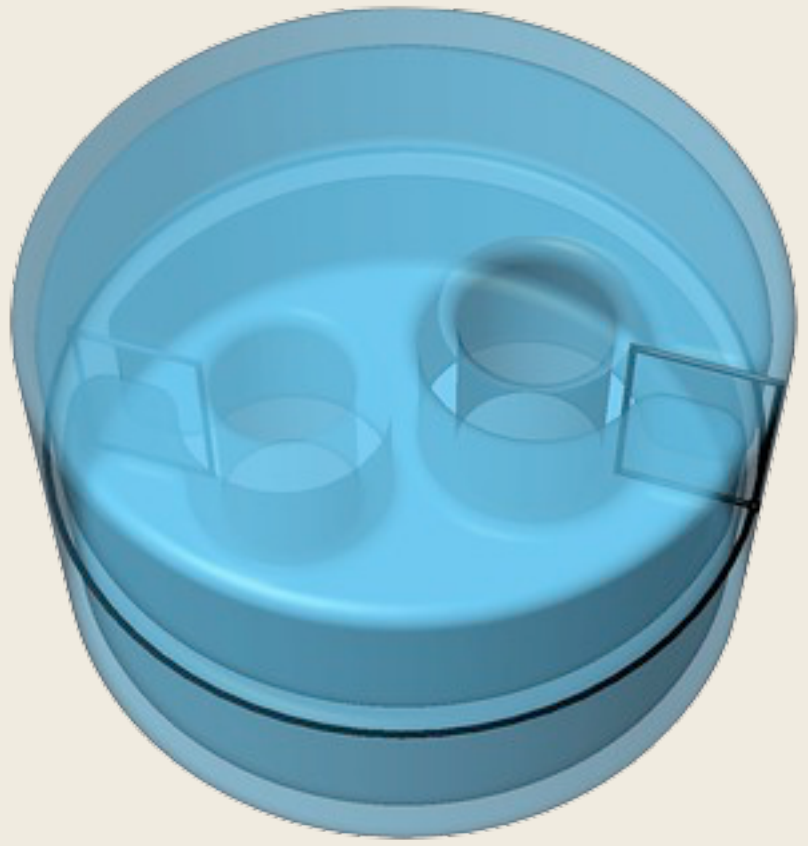
DOES A PERFECT HDVF ALWAYS EXISTS ?

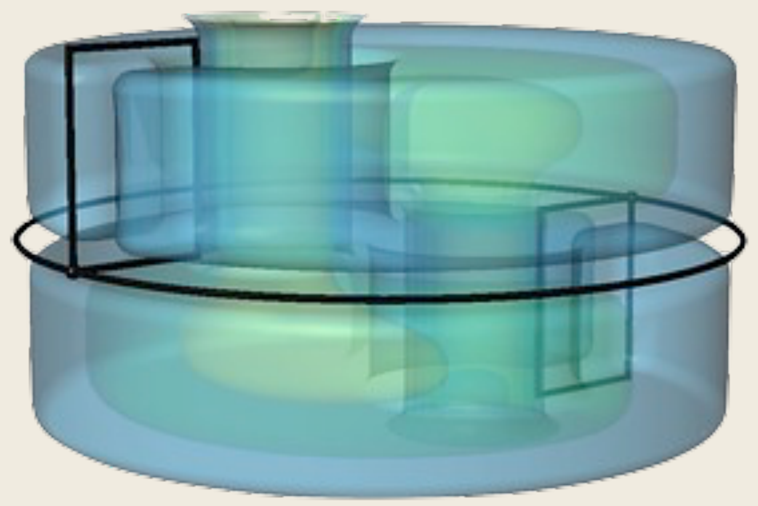


Non perfect DGVF
(but no pair can be added)

Discrete Morse theory
(R. Foreman, 1998)

- Computing an optimal DGVF is **NP-hard in general**
- Some simplicial complexes **do not admit any perfect DGVF**
    - Bing's house (example of contractible but not collapsible complex)
    - Dunce hat





Because of acyclicity condition

Made in Marseille:
Homological Discrete Vector Fields (HDVF)

Phd Aldo Gonzalez Lorenzo

Homological Discrete Vector Fields (allowing cycles)

K (simplicial) complex

- A **HDVF** is a partition:

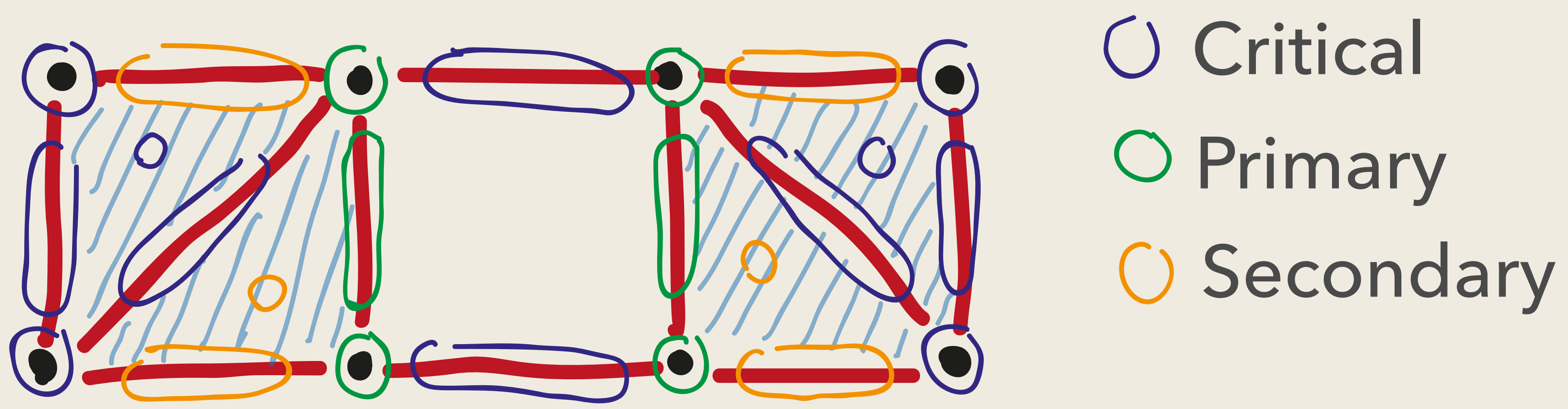
$$K = P \sqcup S \sqcup C$$

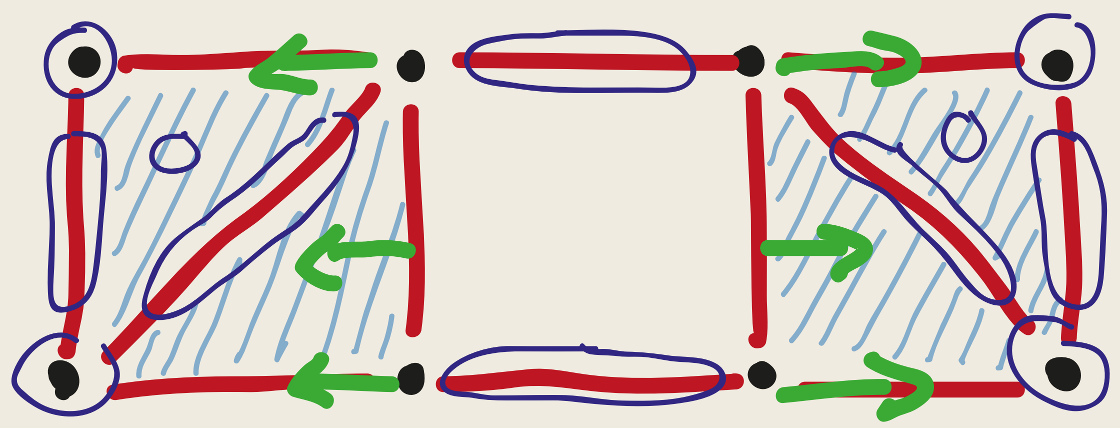
  such that $\partial(S_{q+1})|_{P_q}$ (reduced boundary matrix) is invertible

- A HDVF induces a reduction

A discrete vector field V can be associated to any HDVF:

- P primary cells
- S secondary cells
- C critical cells

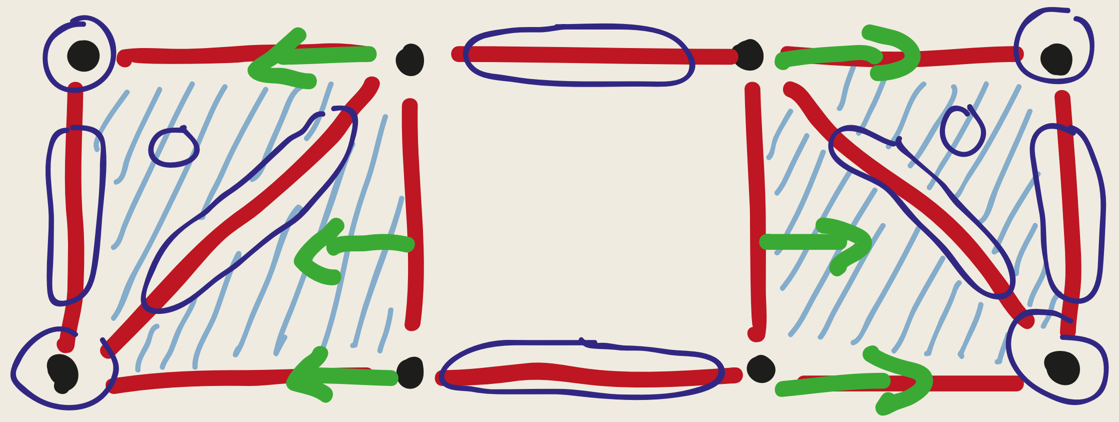




Homological Discrete Vector Fields (allowing cycles)

Associated reduction:

$$h = \begin{array}{c|ccc} & P & S & C \\ \hline P & 0 & 0 & 0 \\ S & H & 0 & 0 \\ C & 0 & 0 & 0 \end{array} \qquad f = \begin{array}{c|ccc} & P & S & C \\ \hline C & F & 0 & I \end{array} \qquad g = \begin{array}{c|c} & C \\ \hline P & 0 \\ S & G \\ C & I \end{array} \qquad \partial' = \begin{array}{c|c} & C \\ \hline C & D \end{array}$$

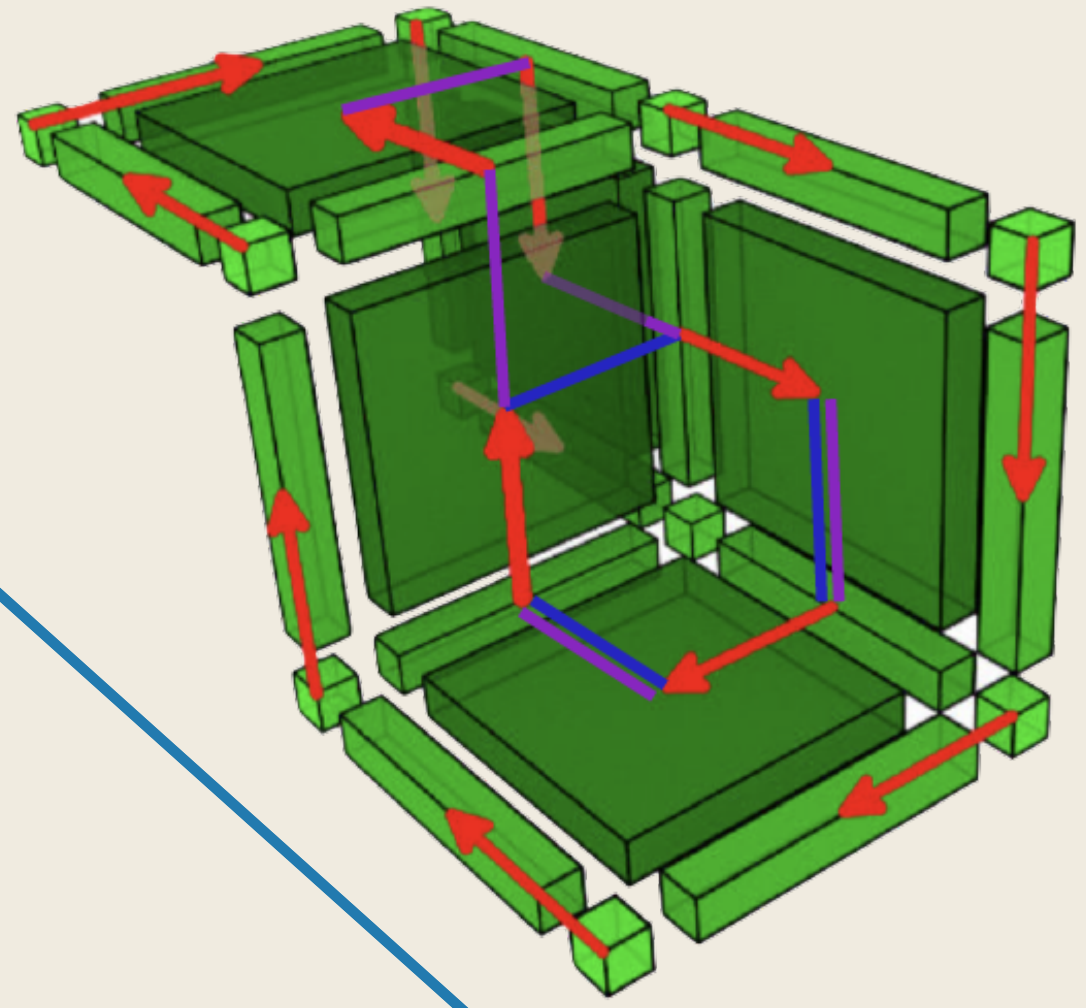


$K = P \sqcup S \sqcup C$

$$H = (\partial(S)|_P)^{-1}$$
$$F = -\partial(S)|_C \times (\partial(S)|_P)^{-1}$$
$$G = -(\partial(S)|_P)^{-1} \times \partial(C)|_P$$
$$D = \partial(C)|_C + F \times \partial(C)|_P = \partial(C)|_C + \partial(S)|_C \times G$$

Reduced complex: critical cells

Homological Discrete Vector Fields (allowing cycles)



Every DGVF is an HDVF

Provides an HDVF

MORSE THEORY

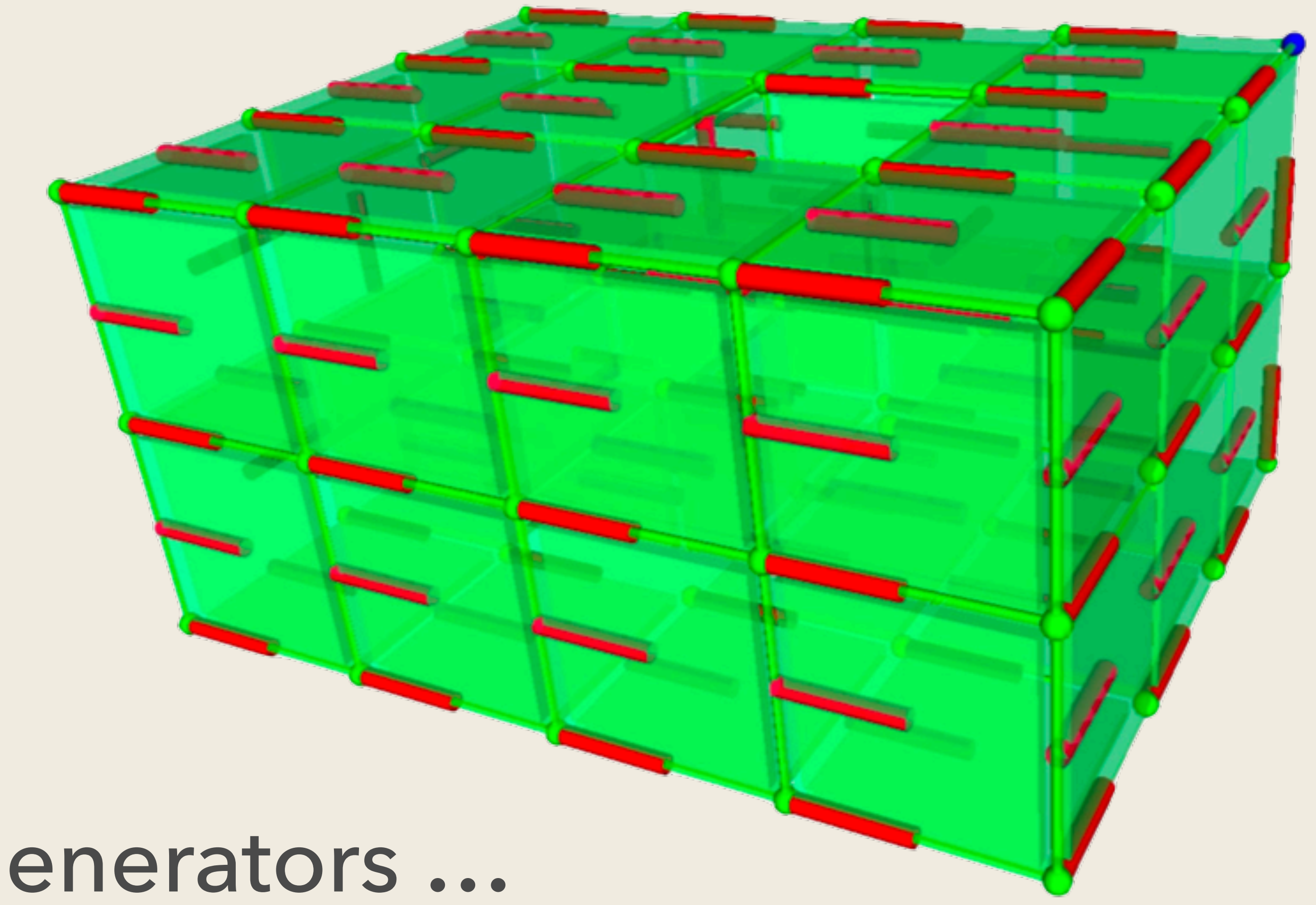
EFFECTIVE HOMOLOGY

SMITH NORMAL FORM

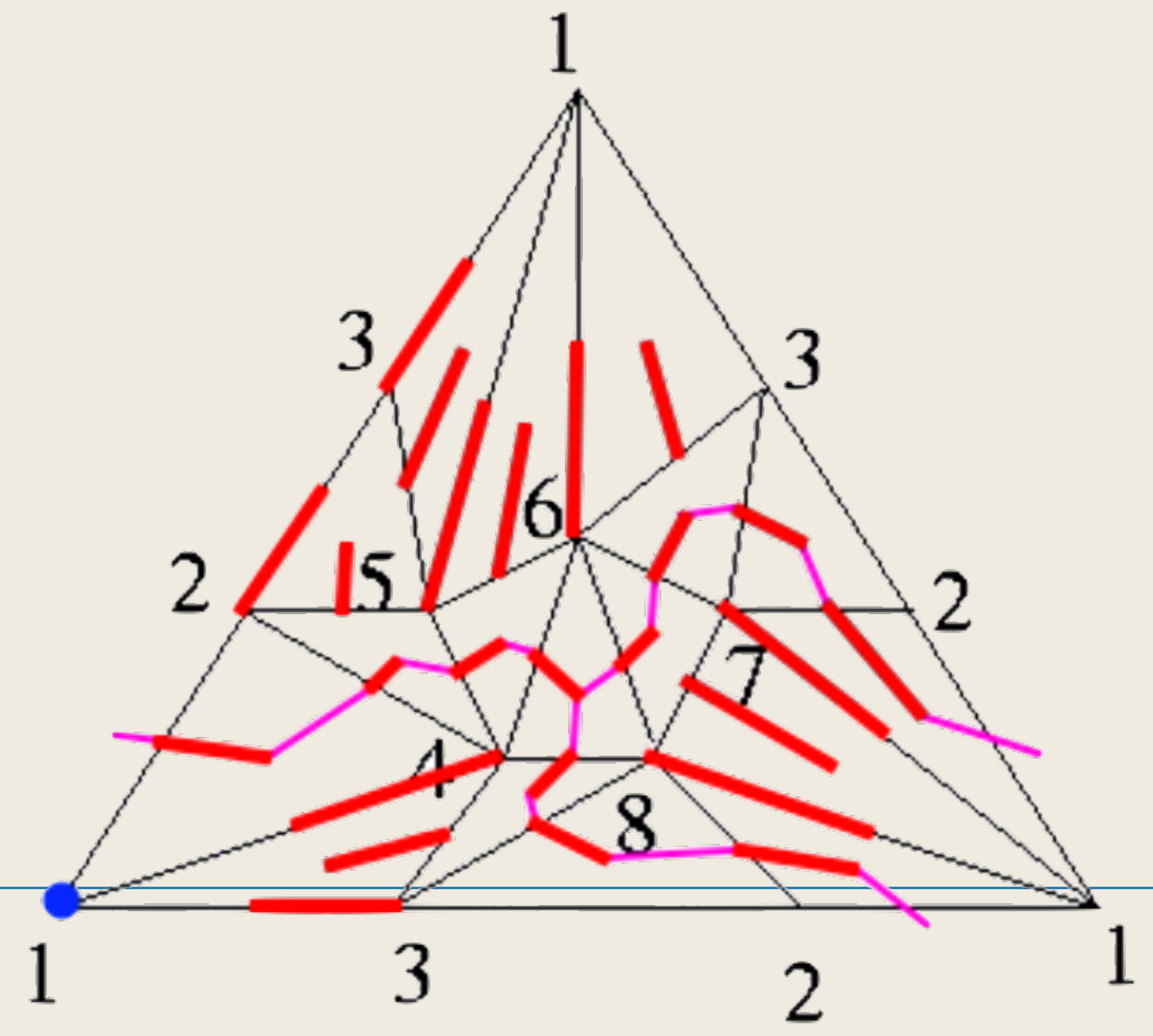
PERSISTENT HOMOLOGY

Homological Discrete Vector Fields (allowing cycles)

- Computing an optimal HDVF has **cubical complexity**
- We **don't know** any simplicial complexe that does not admit a perfect HDVF
  - Bing's house



  - Dunce hat



- Provides homology generators …

Thanks for you attention