

A concrete model of non well-founded linear logic

Thomas Ehrhard's 60th Birthday

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based on joint work with Thomas Ehrhard and Alexis Saurin

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Tarski theorem

Let (X, \leq) be a complete lattice, and F be an increasing function on X . Then the set P of all fixpoints F is a complete lattice.

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$$\mu X.F(X) = \bigcap P = \bigcap \{x \mid F(x) \leq x\}$$

$$\frac{}{F(\mu X.F(X)) \leq \mu X.F(X)} \qquad \frac{F(S) \leq S}{\mu X.F(X) \leq S}$$

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$$\frac{\Delta, F(\nu X.F(X)) \vdash \Gamma}{\Delta, \nu X.F(X) \vdash \Gamma}$$

$$\frac{S \vdash F(S)}{S \vdash \nu X.F(X)}$$

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$\Gamma \vdash \Delta \rightsquigarrow \vdash \Gamma^\perp, \Delta$:

$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} \quad \frac{\vdash S^\perp, F(S)}{\vdash S^\perp, \nu X.F(X)}$$

Cut-elimination fails...

$$\frac{\frac{\overline{\vdash 0, 0, \top}}{\vdash 0, 0, \top} (\top) \quad \frac{\overline{\vdash 0, \top}}{\vdash 0, \nu X.X} (\top)}{\vdash 0, 0, \nu X.X} (\text{cut})$$

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↓

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μLL_∞^1

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$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} (\mu) \quad \frac{\vdash \Gamma, F(\nu X.F(X))}{\vdash \Gamma, \nu X.F(X)} (\nu)$$

+ a possibility to have infinite trees.

Example

$$\text{nat} = \mu X(1 \oplus X)$$

$$\frac{\frac{\frac{\frac{\overline{\quad} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu\text{-fold})}{\vdash \text{nat}} (\perp)}{\vdash \text{nat}, \perp} \quad * \vdash \text{nat}, \text{nat}^\perp}{\frac{\vdash \text{nat}, \perp \& \text{nat}^\perp}{* \vdash \text{nat}, \text{nat}^\perp} (\nu)} (\&)}$$

But...

$$\frac{\frac{\vdots}{\vdash \nu X.X} (\nu) \quad \frac{\vdots}{\vdash \Gamma, \mu X.X} (\mu)}{\vdash \Gamma} (\text{cut})$$

Validity condition

- ▶ An **occurrence** is a formula A together with an address α , denoted as A_α .
- ▶ Extend the usual sub-formula with $\sigma X F \rightarrow_{FL} F(\sigma X F)$ where σ is either ν or μ .
- ▶ B_β is a **FL-sub-occurrence** of A_α if $A \rightarrow_{FL}^* B$ and $\beta \preceq_{sw} \alpha$.
- ▶ A **thread** is a sequence $t = (A_i)_{i \in \omega}$ of **occurrences** such that for all i either A_{i+1} is a **FL-sub-occurrence** of A_i or $A_i = A_{i+1}$.
- ▶ If $t = (A_i)_{i \in \omega}$ is a **thread** we use \bar{t} for the sequence obtained by forgetting the addresses of the occurrences of t .
- ▶ $\text{Inf}(t)$ is the set of formulas that occurs infinitely often in \bar{t} .
- ▶ A **valid thread** t is a non-stationary thread such that $\min(\text{Inf}(t))$ is a ν -formula.
- ▶ A **valid proof** π is a pre-proof π such that for any infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$, there is a non stationary **valid thread** $t = (A_i)_{i > j}$ where $j \in \omega$ and $\forall i > j (A_i \in \Gamma_i)$.

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Example

$F = \mu X.(\nu Y.(X \otimes Y))$ where $G = \nu Y.(F \otimes Y)$.

$$\frac{\frac{\frac{*_2 \vdash F \quad *_1 \vdash G}{\vdash F \otimes G} (\otimes)}{*_1 \vdash G} (\nu)}{*_2 \vdash F} (\mu)}{}$$

Example

$F = \nu X. \mu Y. (1 \oplus (X \wp (Y \oplus \perp)))$ and
 $G = \mu Y. (1 \oplus (F \wp (Y \oplus \perp)))$.

$$\begin{array}{c}
 \frac{* \vdash F, G}{\vdash F, \perp, G} (\perp) \\
 \frac{\vdash F, \perp, G}{\vdash F, G \oplus \perp, G} (\oplus_2) \\
 \frac{\vdash F, G \oplus \perp, G}{\vdash (F \wp (G \oplus \perp)), G} \wp \\
 \frac{\vdash (F \wp (G \oplus \perp)), G}{\vdash 1 \oplus (F \wp (G \oplus \perp)), G} (\oplus_2) \\
 \frac{\vdash 1 \oplus (F \wp (G \oplus \perp)), G}{\vdash G, G} (\mu) \\
 \frac{\vdash G, G}{* \vdash F, G} (\nu)
 \end{array}$$

Totality candidates on a set E

Given $\mathcal{T} \subseteq \mathcal{P}(E)$ we set

$$\mathcal{T}^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$$

Definition (Totality candidates)

\mathcal{T} is a *totality candidate* for E if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

(Equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$, equivalently $\mathcal{T} = \mathcal{S}^\perp$ for some $\mathcal{S} \subseteq \mathcal{P}(E)$.)

Fact

- ▶ \mathcal{T} is a *totality candidate* on E iff $\mathcal{T} \subseteq \mathcal{P}(E)$ and $\mathcal{T} = \uparrow\mathcal{T}$.
- ▶ $\text{Tot}(X)$ (The set of all *totality candidates* on E), ordered with \subseteq , is a *complete lattice* (it is closed under arbitrary intersections).

Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, \mathcal{T}X)$ where

- ▶ $|X|$ is a set
- ▶ $\mathcal{T}X$ is a totality candidate on $|X|$, that is, a \uparrow -closed subset of $\mathcal{P}(|X|)$.

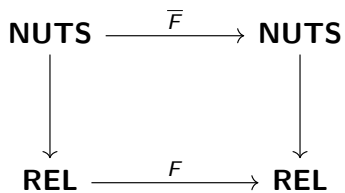
$t \in \mathbf{NUTS}(X, Y)$ if $t \in \mathbf{REL}(|X|, |Y|)$ and

$$\forall u \in \mathcal{T}X \quad t \cdot u \in \mathcal{T}Y$$

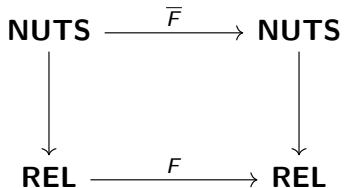
Fact

NUTS is a model of LL where the proofs are interpreted exactly as in **REL**.

Interpretation of $\mu X.F$ in **NUTS**

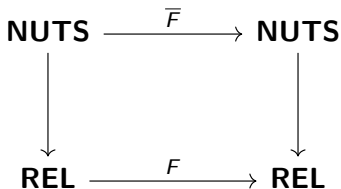


Interpretation of $\mu X.F$ in **NUTS**



$\bar{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathcal{T}(FX)$.

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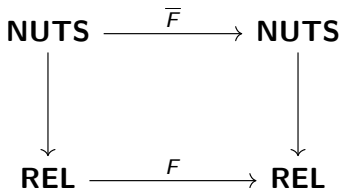
$\bar{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathcal{T}(FX)$.

Assume μF exists.

$$g : \text{Tot}(\mu F) \rightarrow \text{Tot}(\mu F)$$

$$R \mapsto \Phi R$$

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Assume μF exists.

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$$R \mapsto \Phi R$$

By Tarski theorem, μg exists.

$$\mu \bar{F} = (\mu F, \mu g).$$

NUTS as a denotational model of μLL_∞

$$\left[\left[\frac{\vdots \pi}{\vdash \Gamma, F[\mu XF/X]} \right] (\mu) \right] = \llbracket \pi \rrbracket \quad \left[\left[\frac{\vdots \pi}{\vdash \Gamma, F[\nu XF/\zeta]} \right] (\nu) \right] = \llbracket \pi \rrbracket$$

Interpretation of proofs:

$$\llbracket \pi \rrbracket_{\text{REL}} = \bigcup_{\rho \in \text{fin}(\pi)} \llbracket \rho \rrbracket_{\text{REL}}$$

Theorem: If π and π' are μLL_∞ proofs of Γ and π reduces to π' by the cut-elimination rules of μLL_∞ , then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$.

Validity implies totality

Theorem: If π is a valid proof of the sequent $\vdash \Gamma$, then $[[\pi]] \in \mathcal{T}[[\Gamma]]$.

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Theorem: If π is a valid proof of the sequent $\vdash \Gamma$, then $\llbracket \pi \rrbracket \in \mathcal{T}[\llbracket \Gamma \rrbracket]$.

The proof is similar to the proof of soundness of $LKID^\omega$ in ².

We needed to adapt the proof in two aspects:

- ▶ considering μLL_∞ instead of $LKID^\omega$,
- ▶ and deal with the denotational semantics instead of Tarskian semantics.

Adapation for μLL_∞ : somehow done in ³

So, basically, the main point of this proof is adapting a Tarskian soundness theorem to a denotational semantic soundness.

²James Brotherston. Sequent Calculus Proof Systems for Inductive Def-initions. PhD thesis, University of Edinburgh, November 2006.

³Amina Doumane. On the infinitary proof theory of logics with fixedpoints. PhD thesis, Paris Diderot University, 2017.

An example

A syntactic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).

$\llbracket 1 \oplus 1 \rrbracket = (\{(1, \star), (2, \star)\}, \mathcal{T}(\llbracket 1 \oplus 1 \rrbracket))$ where

$$\mathcal{T}(\llbracket 1 \oplus 1 \rrbracket) = \mathcal{P}(\llbracket 1 \oplus 1 \rrbracket) \setminus \emptyset$$

For any proof π of $1 \oplus 1$, we have $\llbracket \pi \rrbracket \in \mathcal{T}(\llbracket 1 \oplus 1 \rrbracket)$.

Hence $\llbracket \pi \rrbracket \neq \emptyset$.