A concrete model of non well-founded linear logic

Thomas Ehrhard's 60th Birthday

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based on joint work with Thomas Ehrhard and Alexis Saurin

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Tarski theorem

Let $(X, \leq)$ be a complete lattice, and F be an increasing function on X. Then the set P of all fixpoints F is a complete lattice.

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$$\mu X.F(X) = \bigcap P = \bigcap \{x \mid F(x) \leq x\}$$

$$\frac{}{F(\mu X.F(X)) \leq \mu X.F(X)} \qquad \frac{F(S) \leq S}{\mu X.F(X) \leq S}$$

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$$\frac{\Delta, F(\nu X.F(X)) \vdash \Gamma}{\Delta, \nu X.F(X) \vdash \Gamma} \qquad \frac{S \vdash F(S)}{S \vdash \nu X.F(X)}$$

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$\Gamma \vdash \Delta \quad \rightsquigarrow \quad \vdash \Gamma^{\perp}, \Delta$:

$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma} \qquad \frac{\vdash S^{\perp}, F(S)}{\vdash S^{\perp}, \nu X.F(X)}$$

Cut-elimination fails...

$$\dfrac{\dfrac{}{\vdash 0, 0, \top}\ (\top) \qquad \dfrac{\dfrac{}{\vdash 0, \top}\ (\top)}{\vdash 0, \nu X.X}}{\vdash 0, 0, \nu X.X}\ (\text{cut})$$

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$\mu\mathrm{LL}_\infty$[1]

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$$\frac{\vdash F(\mu X.F(X)), \Gamma}{\vdash \mu X.F(X), \Gamma}\ (\mu) \qquad \frac{\vdash \Gamma, F(\nu X.F(X))}{\vdash \Gamma, \nu X.F(X)}\ (\nu)$$

$+$ a possibility to have infinite trees.

[1] David Baelde, Amina Doumane, Alexis Saurin: Infinitary Proof Theory: the Multiplicative Additive Case.

Example

$\mathsf{nat} = \mu X(1 \oplus X)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1 \oplus \mathsf{nat}}\,(\oplus_1)}{\vdash \mathsf{nat}}\,(\mu - \mathsf{fold})}{\vdash \mathsf{nat}, \bot}\,(\bot) \qquad * \vdash \mathsf{nat}, \mathsf{nat}^\perp}{\dfrac{\vdash \mathsf{nat}, \bot \,\&\, \mathsf{nat}^\perp}{* \vdash \mathsf{nat}, \mathsf{nat}^\perp}\,(\nu)}\,(\&)
$$

But...

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \nu X.X}\,(\nu)}{\vdash \nu X.X}\,(\nu) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma, \mu X.X}\,(\mu)}{\vdash \Gamma, \mu X.X}\,(\mu)}{}}{\vdash \Gamma}\,(\textbf{cut})$$

Validity condition

- An occurrence is a formula A together with an address α, denoted as A_α.
- Extend the usual sub-formula with $\sigma X\, F \to_{FL} F(\sigma X\, F)$ where σ is either ν or μ.
- B_β is a FL-sub-occurrence of A_α if $A \to_{FL}^\star B$ and $\beta \preceq_{\mathsf{sw}} \alpha$.
- A thread is a sequence $t = (A_i)_{i \in \omega}$ of occurrences such that for all i either A_{i+1} is a FL-sub-occurrence of A_i or $A_i = A_{i+1}$.
- If $t = (A_i)_{i \in \omega}$ is a thread we use $\overline{t}$ for the sequence obtained by forgetting the addresses of the occurrences of t.
- $\mathsf{Inf}(t)$ is the set of formulas that occurs infinitely often in $\overline{t}$.
- A valid thread t is a non-stationary thread such that $\min(\mathsf{Inf}(t))$ is a ν-formula.
- A valid proof π is a pre-proof π such that for any infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$, there is a non stationary valid thread $t = (A_i)_{i > j}$ where $j \in \omega$ and $\forall i > j (A_i \in \Gamma_i)$.

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Example

$F = \mu X.(\nu Y.(X \otimes Y))$ where $G = \nu Y.(F \otimes Y)$.

$$\dfrac{\dfrac{\dfrac{*_2 \vdash F \qquad *_1 \vdash G}{\vdash F \otimes G}\ (\otimes)}{*_1 \vdash G}\ (\nu)}{*_2 \vdash F}\ (\mu)$$

Example

$F = \nu X.\mu Y.(1 \oplus (X \parr (Y \oplus \bot)))$ and
$G = \mu Y.(1 \oplus (F \parr (Y \oplus \bot)))$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{* \vdash F, G}{\vdash F, \bot, G}\,(\bot)}{\vdash F, G \oplus \bot, G}\,(\oplus_2)}{\vdash (F \parr (G \oplus \bot)), G}\,\parr}{\vdash 1 \oplus (F \parr (G \oplus \bot)), G}\,(\oplus_2)}{\vdash G, G}\,(\mu)}{* \vdash F, G}\,(\nu)
$$

Totality candidates on a set E

Given $\mathcal{T} \subseteq \mathcal{P}(E)$ we set

$$\mathcal{T}^{\perp} = \left\{ u' \subseteq E \;\middle|\; \forall u \in \mathcal{T}\ u \cap u' \neq \varnothing \right\}$$

Definition (Totality candidates)

$\mathcal{T}$ is a *totality candidate* for E if $\mathcal{T} = \mathcal{T}^{\perp\perp}$.

(Equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$, equivalently $\mathcal{T} = \mathcal{S}^{\perp}$ for some $\mathcal{S} \subseteq \mathcal{P}(E)$.)

Fact

- *$\mathcal{T}$ is a totality candidate on E iff $\mathcal{T} \subseteq \mathcal{P}(E)$ and $\mathcal{T} = \uparrow\mathcal{T}$.*
- $\mathrm{Tot}(X)$ *(The set of all totality candidates on E), ordered with $\subseteq$, is a complete lattice (it is closed under arbitrary intersections).*

Non-uniform totality spaces (NUTS)

A NUTS is a pair $X = (|X|, \mathcal{T}X)$ where

- $|X|$ is a set
- $\mathcal{T}X$ is a totality candidate on $|X|$, that is, a $\uparrow$-closed subset of $\mathcal{P}(|X|)$.

$t \in \mathbf{NUTS}(X, Y)$ if $t \in \mathbf{REL}(|X|, |Y|)$ and

$$\forall u \in \mathcal{T}X \quad t \cdot u \in \mathcal{T}Y$$

Fact

NUTS *is a model of LL where the proofs are interpreted exactly as in* **REL***.*

Interpretation of $\mu X.F$ in **NUTS**

$$\begin{array}{ccc} \mathbf{NUTS} & \xrightarrow{\quad \overline{F} \quad} & \mathbf{NUTS} \\ \downarrow & & \downarrow \\ \mathbf{REL} & \xrightarrow{\quad F \quad} & \mathbf{REL} \end{array}$$

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$\overline{F} : (X, U) \mapsto (FX, \Phi U)$ where $\Phi U \in \mathcal{T}(FX)$.

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Assume μF exists.

$$g : \mathrm{Tot}(\mu F) \to \mathrm{Tot}(\mu F)$$

$$R \mapsto \Phi R$$

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By Tarski theorem, μg exists.

$$\mu\overline{F} = (\mu F, \mu g).$$

NUTS as a denotational model of $\mu\mathrm{LL}_\infty$

$$\left[\!\!\left[\dfrac{\begin{array}{c} \vdots\ \pi \\ \vdash \Gamma, F[\mu X F/X] \end{array}}{\vdash \Gamma, \mu X F}\ (\mu) \right]\!\!\right] = [\![\pi]\!] \qquad \left[\!\!\left[\dfrac{\begin{array}{c} \vdots\ \pi \\ \vdash \Gamma, F[\nu X F/\zeta] \end{array}}{\vdash \Gamma, \nu Y F}\ (\nu) \right]\!\!\right] = [\![\pi]\!]$$

Interpretation of proofs:

$$[\![\pi]\!]_{\mathbf{REL}} = \bigcup_{\rho \in \mathsf{fin}(\pi)} [\![\rho]\!]_{\mathbf{REL}}$$

Theorem: If π and π' are $\mu\mathrm{LL}_\infty$ proofs of Γ and π reduces to π' by the cut-elimination rules of $\mu\mathrm{LL}_\infty$, then $[\![\pi]\!] = [\![\pi']\!]$.

Validity implies totality

Theorem: If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}[\![\Gamma]\!]$.

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The proof is similar to the proof of soundness of LKID^ω in [2].

We needed to adapt the proof in two aspects:

- considering $\mu\mathrm{LL}_\infty$ instead of $LKID^\omega$,
- and deal with the denotational semantics instead of Tarskian semantics.

Adapation for $\mu\mathrm{LL}_\infty$: somehow done in [3]

So, basically, the main point of this proof is adapting a Tarskian soundness theorem to a denotational semantic soundness.

[2] James Brotherston.Sequent Calculus Proof Systems for Inductive Def-initions. PhD thesis, University of Edinburgh, November 2006.

[3] Amina Doumane. On the infinitary proof theory of logics with fixedpoints. PhD thesis, Paris Diderot University, 2017.

An example

A syntatic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).
$[\![1 \oplus 1]\!] = (\{(1, \star), (2, \star)\}, \mathcal{T}[\![1 \oplus 1]\!])$ where

$$\mathcal{T}([\![1 \oplus 1]\!]) = \mathcal{P}(|[\![1 \oplus 1]\!]|) \backslash \emptyset$$

For any proof π of $1 \oplus 1$, we have $[\![\pi]\!] \in \mathcal{T}[\![1 \oplus 1]\!]$.
Hence $[\![\pi]\!] \neq \emptyset$.