Condorcet meets Gustave (revisited)

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for

Thomas Ehrhard's 60th birthday

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A TALK ABOUT (NON) SEQUENTIALITY

what else ?

Largely inspired from G. Huet's talk at GGJJ2011

with some additions of my own

Thanks to Antonio Bucciarelli for discussions

Once upon a time in the late 1960's - early 1970's

- Scott domains and Scott continuity.

For our purpose $Bool_\perp = \{\perp, T, F\}$ $\quad 1_\perp = \{\perp, T\}$

$$\begin{array}{ccc} T & & F \\ & \diagdown \; \diagup & \\ & \perp & \end{array} \qquad \begin{array}{c} T \\ | \\ \perp \end{array}$$

Our playground: monotone (= continuous here) functions

$$f : (Bool_\perp)^n \to 1_\perp$$

Such functions can be recovered by their trace: the set of minimal points $(x_1 \dots x_n)$ s.t. $f(x_1 \dots x_n) = T$ and hence can be described by an $m \times n$-matrix (m = number of minimal points)

$$\begin{pmatrix} x_1^1 & \dots & x_n^1 \\ \vdots & & \vdots \\ x_1^m & & x_n^m \end{pmatrix}$$

- Scott noticed the problematic behaviour of "por" $: (Bool_\perp)^2 \to 1_\perp$ given by

$$\begin{pmatrix} \perp & T \\ T & \perp \end{pmatrix}$$

This launched the quest for a (genuine) fully abstract (FA) model of PCF.

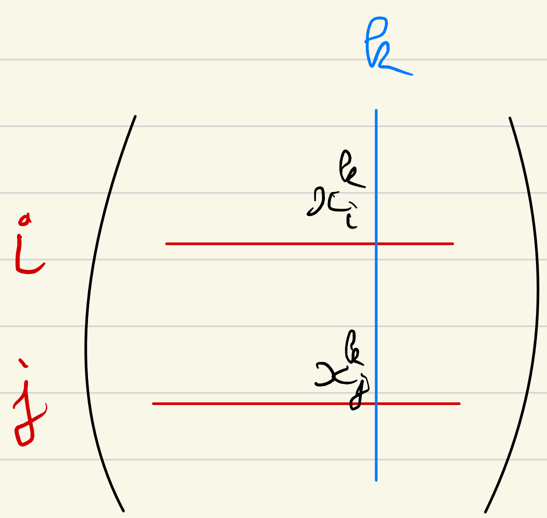
Once upon a time in the late 1970's

- Berry (= Gustave, see gallium.inria.fr/~huet/PUBLIC/GGJJ.pdf)

"killed" por by requiring all pairs of minimal points to be incompatible = stable functions (rediscovered by Girard in early 1980's $\rightsquigarrow$ linear logic)

In the present setting, in terms of matrices:

$$\forall i,j\ \exists k\quad x_i^k = T \text{ and } x_j^k = F \text{ or } x_i^k = F \text{ and } x_j^k = T$$



Indeed "por" is not stable:

$$\begin{pmatrix} \bot & T \\ T & \bot \end{pmatrix} \qquad \left.\begin{matrix} (\bot, T) \\ (T, \bot) \end{matrix}\right\} \leq (T, T)$$

- But Berry noticed the problematic character of the function

$$\text{GUSTAVE} = \begin{pmatrix} \bot & T & F \\ F & \bot & T \\ T & F & \bot \end{pmatrix}$$

Problematic = non sequential

In both cases ("por" and GUSTAVE), problematic means not sequential (at $(\perp,\perp,\perp)$)

- A function $f : (Bool_\perp)^n \to 1_\perp$ is sequential if

$$\exists i \, \left(f(x_1 \dots x_n) = T \right) \Rightarrow x_i \neq \perp$$

So that we can start to program f by writing

if $x_i = T$ then ...

- Sequential $\Rightarrow$ stable

Summary so far

- por is continuous but not stable
- GUSTAVE is stable but not sequential

- Then all full abstraction hunters searched ways to "kill" GUSTAVE, and Thomas is one of them, who did it (with Antonio Bucciarelli) via strong stability.
- But in this talk, we love GUSTAVE!

Definition (Huet 2011) A Gustave function is a function $f : (\mathrm{Bool}_\perp)^n \to \mathbb{1}_\perp$ that is

- stable, and
- not sequential at $(\perp, \perp, \perp)$

Equivalent definition: each column of the matrix contains a $\perp$:

$$\underbrace{\left.\begin{pmatrix} \perp & \cdots & & \\ \cdot & \perp & \cdots & \\ \cdots & & & \perp \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \end{pmatrix}\right\} m}_{n}$$

(in particular, $m \geq n$)

(up to permutation of lines)

Once upon a time in 1785 in the kingdom of France

ESSAI
SUR L'APPLICATION
DE L'ANALYSE
À LA
PROBABILITÉ
DES DÉCISIONS
Rendues à la pluralité des voix.

Par M. LE MARQUIS DE CONDORCET, Secrétaire perpétuel de l'Académie des Sciences, de l'Académie Françoiſe, de l'Inſtitut de Bologne, des Académies de Péterſbourg, de Turin, de Philadelphie & de Padoue.

Quòd ſi deficiant vires audacia certè
Laus erit, in magnis & voluiſſe ſat eſt.

A PARIS,
DE L'IMPRIMERIE ROYALE.
M. DCCLXXXV.

Condorcet's (1743-1794) voting procedure

We have
- a set $C = \{c_1, \dots, c_n\}$ of **candidates**
- a set $V = \{v_1, \dots, v_{2N+1}\}$ of **voters**

Each voter gives a total order of preference of all candidates

Thus the data of a Condorcet ballot consists in a function

$\varphi : V \to \mathrm{Bij}(C, \{1, \dots, n\})$ (the set of bijections from C to $\{1, \dots n\}$)

We write $c <_v c'$ if $\varphi(v)(c) < \varphi(v)(c')$

"v ranks c' above c"

We write $c <_V c'$ if $\mathrm{card}\{v \mid c <_v c'\} > N$

"a majority of voters ranks c' above c"

c' **beats** c

(Strictly speaking, we should write $c <_v^{\varphi} c'$, $c <_V^{\varphi} c'$)

Observe that for all c, c' we have (exclusively)

$$\boxed{c = c' \quad \text{or} \quad c <_V c' \quad \text{or} \quad c' <_V c}$$

also written $c >_V c'$

i.e., $<_V$ is a total strict relation.

Condorcet paradox

Warning. Unlike $<_v$, $<_V$ may not be transitive. The basic example is

$$\boxed{\begin{array}{l} C = \{A, B, C\}, \quad V = \{1, 2, 3\} \\ \varphi(1) = A > B > C \\ \varphi(2) = B > C > A \\ \varphi(3) = C > A > B \end{array}}$$

We have $A >_V B$, $B >_V C$, $C >_V A$
(supported each by two voters over three)

Hence $A >_V C$ does not hold!

If $<_V$ admits a (necessarily unique) maximum, i.e. c s.t. $\forall c' \neq c \;\; c >_V c'$, then c is called Condorcet winner.

There is no winner in this example. The possible absence of a winner is called Condorcet paradox.

Note. Lack of transitivity does not prevent the existence of a winner. Just add a candidate D and put

$$\varphi(1) = D > A > B > C$$
$$\varphi(2) = D > B > C > A$$
$$\varphi(3) = D > C > A > B$$

Then $<_V$ is not transitive, but D wins.

We have
- $<_V$ transitive $\Rightarrow \exists$ winner
- for $\mathrm{card}(C) = 3$, the converse holds
- for $\mathrm{card}(C) < 3$ there is a winner

Abstracting from the set of voters

Without loss of generality (as we shall see), we abstract from V and consider a total strict relation $<$ on a set $C = \{c_1, \dots, c_n\}$ of candidates. The relation $<$ can be represented as a $n \times n$-matrix $M_<$ with entries in $\mathrm{Bool}_\perp$, setting

$$M_{ij} = \begin{cases} T & \text{if } c_j < c_i \\ \perp & \text{if } i = j \\ F & \text{if } c_i < c_j \end{cases}$$

For example, the matrix associated to the counterexample above is

$$\begin{pmatrix} \perp & T & F \\ F & \perp & T \\ T & F & \perp \end{pmatrix}$$

We've met before...

The matrices $M_<$ have the following features:

- They are square matrices $(n \times n)$
- The diagonal is filled with $\perp$, and these are the only $\perp$ entries
- Setting $\overline{T} = F$, $\overline{F} = T$ and $\overline{\perp} = \perp$, we have $\forall i,j \quad c_{ji} = \overline{c_{ij}}$

<u>Definition</u> (PLC2022). Such a matrix is called <u>semi-Condorcet</u>

To win or not to win

born in 1470

not

- As the seigneur de La Palice could not have said, it is supereasy to detect the winner, if any, on a semi-Condorcet matrix: find a (necessarily unique) line i of the form

$$\top \ldots \top \perp \top \ldots \top \qquad (c_{ij} = \top \text{ for all } j \neq i)$$

- But there is another characterisation that gives more information in the case of the absence of a winner: there is a uniquely determined core subset X of candidates that cause (via $\angle$) the paradox.

(In terms of voters, X is minimum such that if we erase the candidates in $C \setminus X$ from the voters, keeping the rest of the expressed preferences, then there is no winner)

Characterising the absence of a winner

Definition (Huet 2011). We define the relation $\prec^T$ as follows:

$c \prec^T c'$ if
- $c < c'$ and
- for all $c'' \neq c, c'$, if $c'' < c$, then $c'' < c$

c' defeats c "c' beats c, and all candidates that c beats"

Lemma. The relation $\prec^T$ is transitive.

Algorithm
- Initialisation $X := C$
- while $\exists c, c' \in X$ such that $c' \prec^T_X c$, set $X := X - \{c'\}$

w.r.t X! Slowly, take $<_X$ = restriction of $<$ to X, and then $\prec^T_X = (<_X)^T$

Proposition (Huet 2011)
- Given C and $<$ total strict on C,

  there is a winner (i.e. a maximum for $<$)

  $\Updownarrow$

  the algorithm terminates with $X = \{c_i\}$ for some i ← the winner!
- The final state X reached by the algorithm is independent of the choices made by the algorithm (choice of c at each iteration)
- If there is no winner, the machine stops at some X with $\mathrm{card}(X) \geq 3$, and $M_{<_X}$ displays a Gustave function.

We call $M_{<_X}$ the Condorcet witness of $<$.

(and $\mathrm{card}(X) \neq 4$ (PLC 2022), see below)

Applying the algorithm

Algorithm

- Initialisation $X := C$
- while $\exists\, c, c' \in X$ such that $c' <^{T}_{X} c$, set $X := X - \{c'\}$

- With a winner

$$\begin{array}{cccc} c_1 & \perp & T & T \\ c_2 & F & \perp & F \\ c_3 & \cancel{F} & \cancel{T} & \cancel{\perp} \end{array} \qquad \begin{array}{ccc} c_1 & \perp & T \\ c_2 & \cancel{F} & \cancel{\perp} \end{array} \qquad \begin{array}{cc} c_1 & \perp \end{array}$$

- Without a winner (obtained by reverse engineering!)

$$\begin{array}{cccccc} c_1 & \perp & T & F & T & F \\ c_2 & F & \perp & T & F & T \\ c_3 & T & F & \perp & T & T \\ c_4 & F & T & F & \perp & F \\ c_5 & \cancel{T} & \cancel{F} & \cancel{F} & \cancel{T} & \cancel{\perp} \end{array} \qquad \begin{array}{ccccc} c_1 & \perp & T & F & T \\ c_2 & F & \perp & T & F \\ c_3 & T & F & \perp & T \\ c_4 & \cancel{F} & \cancel{T} & \cancel{F} & \cancel{\perp} \end{array} \qquad \begin{array}{cccc} c_1 & \perp & T & F \\ c_2 & F & \perp & T \\ c_3 & T & F & \perp \end{array}$$

Final: no winner

Proof of the soundness and completeness of Huet's algorithm

Algorithm
- Initialisation $X := C$
- while $\exists c, c' \in X$ such that $c' <^T_X c$, set $X := X - \{c'\}$

Proof of the proposition.

- If there is a winner c_i, it a fortiori defeats all candidates, hence taking $C \setminus \{c_i\} = \{c_{i_1} \dots c_{i_{n-1}}\}$ in any order we can set $X := C \setminus \{c_{i_1}\}$, $X := C \setminus \{c_{i_1}, c_{i_2}\}$ etc. until $X := C \setminus \{c_{i_1}, \dots c_{i_{n-1}}\} = \{c_i\}$
- The converse is an immediate consequence of the following claim:

If [1] $X \subsetneq C$, $c' \in C \setminus X$, $c_1, c_2 \in X$

[2] $c_1 \in X$ wins relatively to X

[3] c_2 defeats c' relatively to $X \cup \{c'\}$

then c wins relatively to $X \cup \{c'\}$.

Proof of the claim. We have

- $c_1 >^T_X c_2$ by [2], or equivalently $c_1 >^T_{X \cup \{c'\}} c_2$
- Hence $\underbrace{c_1 >^T_{X \cup \{c'\}} c_2 >^T_{X \cup \{c'\}} c'}_{3} \underset{\text{transitivity lemma}}{\Rightarrow} c_1 >^T_{X \cup \{c'\}} c' \Rightarrow c_1 > _{X \cup \{c'\}} c'$

Which Gustave functions arise as Condorcet witnesses?

By construction, a Condorcet witness is a (sub) set X of candidates such that $\nexists\, c_i, c_j \in X\; c_i >'_X c_j$, or, equivalently

$$\boxed{\forall c_i, c_j \left(c_i >_X c_j \Rightarrow \exists k \neq i,j \left(c_j >_X c_k \text{ and } c_i <_X c_k\right)\right)}$$

Translating this in terms of matrices, we arrive at:

Definition. A semi-Condorcet matrix M is called a Condorcet matrix iff the following additional condition holds:

- $\forall i,j$, if $M_{ij} = T$ (and hence $M_{ji} = F$), then there exists $k \neq i,j$ s.t. $M_{ik} = F$ and $M_{jk} = T$. Such a k is called a Condorcet witness (of stability of the associated function) for i,j

$$\begin{array}{cc}
 & \begin{array}{ccccccc} \cdots & i & \cdots & j & \cdots & k & \cdots \end{array} \\
\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} &
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \bot & \cdots & T & \cdots & F & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & F & \cdots & \bot & \cdots & T & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{array}$$

A function is called a Condorcet function if its matrix is Condorcet

Thus, by definition, $\text{Condorcet} \subseteq \text{Gustave}$

The eight 3x3 Gustave functions

$$\cancel{\begin{matrix} \bot & F & F \\ T & \bot & T \\ F & T & \bot \end{matrix}} \qquad \cancel{\begin{matrix} \bot & F & F \\ F & \bot & T \\ T & T & \bot \end{matrix}} \qquad \boxed{\begin{matrix} \bot & F & T \\ T & \bot & F \\ F & T & \bot \end{matrix}} \qquad \cancel{\begin{matrix} \bot & F & T \\ F & \bot & F \\ T & T & \bot \end{matrix}}$$

$$\cancel{\begin{matrix} \bot & T & F \\ T & \bot & T \\ F & F & \bot \end{matrix}} \qquad \boxed{\begin{matrix} \bot & T & F \\ F & \bot & T \\ T & F & \bot \end{matrix}} \qquad \cancel{\begin{matrix} \bot & T & T \\ F & \bot & F \\ T & F & \bot \end{matrix}} \qquad \cancel{\begin{matrix} \bot & T & T \\ T & \bot & F \\ F & F & \bot \end{matrix}}$$

- Each of the six excluded functions is refuted for two reasons:
  - not semi-Condorcet
  - no Condorcet witness of stability

This is special to $n=3$!

- The remaining ones are GUSTAVE and its mirror

Gégé (= Gustave = GB) could not have been closer to the marquis de Condorcet!

Summary.

for $n=3$

Condorcet = semi-Condorcet = {GUSTAVE, co-GUSTAVE} $\subsetneq$ Gustave

A curiosity: there is no 4x4 Condorcet function

$$\begin{array}{cccc} \perp & a & b & c \\ \bar a & \perp & d & e \\ \bar b & \bar d & \perp & f \\ \bar c & \bar e & \bar f & \perp \end{array}$$

1 $a=T$, $b=F$, $d=T$

$$\begin{array}{cccc} \perp & T & F & c \\ F & \perp & T & e \\ T & F & \perp & f \\ \bar c & \bar e & \bar f & \perp \end{array}$$

1.1 $e=F$

$$\begin{array}{cccc} \perp & T & F & c \\ F & \perp & T & F \\ T & F & \perp & f \\ \bar c & T & \bar f & \perp \end{array} \quad \text{forces } \bar f = F \qquad \begin{array}{cccc} \perp & T & F & c \\ F & \perp & T & F \\ T & F & \perp & T \\ \bar c & T & F & \perp \end{array}$$

compatible! NO

1.2 $e=T$

$$\begin{array}{cccc} \perp & T & F & c \\ F & \perp & T & T \\ T & F & \perp & f \\ \bar c & F & \bar f & \perp \end{array} \quad \text{forces } \bar c = T \qquad \begin{array}{cccc} \perp & T & F & F \\ F & \perp & T & T \\ T & F & \perp & f \\ T & F & \bar f & \perp \end{array}$$

compatible! NO

2 $a=T$, $c=F$, $e=T$

$$\begin{array}{cccc} \perp & T & b & F \\ F & \perp & d & T \\ \bar b & \bar d & \perp & f \\ T & F & \bar f & \perp \end{array}$$

2.1 $f=T$

$$\begin{array}{cccc} \perp & T & b & F \\ F & \perp & d & T \\ \bar b & \bar d & \perp & T \\ T & F & F & \perp \end{array} \quad \text{forces } \bar b = F \qquad \begin{array}{cccc} \perp & T & T & F \\ F & \perp & d & T \\ F & \bar d & \perp & T \\ T & F & F & \perp \end{array}$$

compatible! NO

2.2 $f=F$

$$\begin{array}{cccc} \perp & T & b & F \\ F & \perp & d & T \\ \bar b & \bar d & \perp & F \\ T & F & T & \perp \end{array} \quad \text{forces } \bar d = T \qquad \begin{array}{cccc} \perp & T & b & F \\ F & \perp & F & T \\ \bar b & T & \perp & F \\ T & F & T & \perp \end{array}$$

compatible! NO

3 $a=F$, $b=T$, $d=F$

$$\begin{array}{cccc} \perp & F & T & c \\ T & \perp & F & e \\ F & T & \perp & f \\ \breve c & \breve e & \bar f & \perp \end{array}$$

3.1 $f=T$

$$\begin{array}{cccc} \perp & F & T & c \\ T & \perp & F & e \\ F & T & \perp & T \\ \breve c & \breve e & F & \perp \end{array} \quad \text{forces } \bar c = T \qquad \begin{array}{cccc} \perp & F & T & F \\ T & \perp & F & e \\ F & T & \perp & T \\ T & \breve e & F & \perp \end{array}$$

compatible! NO

3.2 $f=F$

$$\begin{array}{cccc} \perp & F & T & c \\ T & \perp & F & e \\ F & T & \perp & F \\ \breve c & \breve e & T & \perp \end{array} \quad \text{forces } \bar e = F \qquad \begin{array}{cccc} \perp & F & T & c \\ T & \perp & F & T \\ F & T & \perp & F \\ \breve c & F & T & \perp \end{array}$$

compatible! NO

4 $a=F$, $c=T$, $e=F$

$$\begin{array}{cccc} \perp & F & b & T \\ T & \perp & d & F \\ \bar b & \bar d & \perp & f \\ F & T & \bar f & \perp \end{array}$$

4.1 $f=T$

$$\begin{array}{cccc} \perp & F & b & T \\ T & \perp & d & F \\ \bar b & \bar d & \perp & T \\ F & T & F & \perp \end{array} \quad \text{forces } \bar d = F \qquad \begin{array}{cccc} \perp & F & b & T \\ T & \perp & T & F \\ \bar b & F & \perp & T \\ F & T & F & \perp \end{array}$$

compatible! NO

4.2 $f=F$

$$\begin{array}{cccc} \perp & F & b & T \\ T & \perp & d & F \\ \bar b & \bar d & \perp & F \\ F & T & T & \perp \end{array} \quad \text{forces } \hat b = T \qquad \begin{array}{cccc} \perp & F & F & T \\ T & \perp & d & F \\ T & \bar d & \perp & F \\ F & T & T & \perp \end{array}$$

compatible! NO

Gustave vs semi-Condorcet vs Condorcet for $n \leq 4$

For all n, we have

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \text{semi-Condorcet} \\
\emptyset \subseteq \text{Condorcet} \arrow[urr, phantom, "\subseteq" sloped] \arrow[drr, phantom, "\subseteq" sloped] & & \\
 & & \text{Gustave}
\end{tikzcd}
$$

- $n=2$ $\quad$ $\text{Condorcet} = \text{Gustave} = \emptyset \subsetneq \text{semi-Condorcet} = \left\{ \begin{pmatrix} \bot & T \\ F & \bot \end{pmatrix}, \begin{pmatrix} \bot & F \\ T & \bot \end{pmatrix} \right\}$
- $n=3$

$\text{Condorcet} = \text{semi-Condorcet} = \{\text{GUSTAVE}, \text{co-GUSTAVE}\} \subsetneq \text{Gustave}$

- $n=4$

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \text{Gustave} & \begin{matrix} \bot & T & F & F \\ F & \bot & T & F \\ F & F & \bot & T \\ T & F & T & \bot \end{matrix} \arrow[l, squiggly] \\
\text{Condorcet} = \emptyset \arrow[urr, phantom, "\subsetneq" sloped] \arrow[drr, phantom, "\subsetneq" sloped] & & & \\
 & & \text{semi-Condorcet} &
\end{tikzcd}
$$

$$
\begin{matrix} \bot & T & F & F \\ F & \bot & T & F \\ T & F & \bot & F \\ T & T & T & \bot \end{matrix}
$$

An example of a 5x5 Condorcet function

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

$$\begin{matrix} \perp & T & F & T & F \\ F & \perp & T & F & F \\ T & F & \perp & T & T \\ F & T & F & \perp & T \\ T & T & F & F & \perp \end{matrix}$$

Synthesising a 6x6 Condorcet function

$$\begin{array}{cccccc}
\perp & T & F & T & F & T \\
F & \perp & T & F & F & \\
T & F & \perp & T & T & \\
F & T & F & \perp & T & \\
T & T & F & F & \perp & F \\
F & & & & T & \perp
\end{array}
\qquad
\begin{array}{cccccc}
\perp & T & F & T & F & T \\
F & \perp & T & F & F & T \\
T & F & \perp & T & T & \\
F & T & F & \perp & T & \\
T & T & F & F & \perp & F \\
F & F & & & T & \perp
\end{array}$$

$$\begin{array}{cccccc}
\perp & T & F & T & F & T \\
F & \perp & T & F & F & T \\
T & F & \perp & T & T & F \\
F & T & F & \perp & T & \\
T & T & F & F & \perp & F \\
F & F & T & & T & \perp
\end{array}
\qquad
\begin{array}{cccccc}
\perp & T & F & T & F & T \\
F & \perp & T & F & F & T \\
T & F & \perp & T & T & F \\
F & T & F & \perp & T & F \\
T & T & F & F & \perp & F \\
F & F & T & T & T & \perp
\end{array}$$

Question.

Does

$\emptyset \neq$ Condorcet $\subsetneq$ Semi-Condorcet $\subsetneq$ Gustave

hold for all $n \geq 5$?

[In particular, are there Condorcet functions for all $n \neq 1, 2, 4$?)

Back to voting

Proposition (PLC 2022) For any set $C = \{c_1, \dots, c_n\}$ of candidates, for any total strict relation $<$ on C, there exists a set of voters (of odd cardinality) and a Condorcet ballot such that $<_V = <$.

Lemma For fixed C, if (V, φ) is a Condorcet ballot, then, taking $W = (V.1) \cup (V.2) \cup \{*\}$ and ψ defined as follows:

- $\psi(v.1) = \varphi(v)$
- $\psi(v.2) = \varphi(v)$
- $\psi(*) =$ whatever choice

we have $<_V = <_W$.

Proof. We have, for all $c, c' \in C$:

$$\left| \mathrm{card}\{w \mid w \in W \setminus \{*\} \text{ and } c <_w c'\} - \mathrm{card}\{w \mid w \in W \setminus \{*\} \text{ and } c' <_w c\} \right| \geq 2$$

And hence the single vote of $*$ cannot make a difference.

Proof of Proposition. We proceed by induction on the number of candidates. Assume we have found V such that $<_V =$ restriction of $<$ to $X \subsetneq C$, and pick $c \in C \setminus X$. Then take W, ψ as in the lemma (relatively to X),

- extending ψ on $W \setminus \{*\}$ by setting
  - every $v.1$ places c in first position
  - every $v.2$ places c in last position
- constraining the choice of the vote of $*$ so as to have
  - $*$ prefers c to all candidates c' such that $c' < c$
  - $*$ prefers all candidates c' such that $c' < c$ to c

Once upon a time in the late 1980's

CHAPEAU
BAS À
THOMAS!