To finiteness spaces and beyond!

Michele Pagani, Christine Tasson & Lionel Vaux Auclair
(not necessarily in that order)

Workshop in Honour of Thomas Ehrhard's 60th Birthday,
29–30 September 2022, Paris

The relational model of linear logic

$$|A \otimes B| = |A| \times |B| \quad |A^\perp| = |A| \quad |!A| = \mathfrak{M}_f(|A|) \qquad [\![\pi : \Gamma]\!] \subseteq |\Gamma|$$

$$(a', a) \in \left[\!\!\left[\frac{}{A^\perp, A}\ (ax)\right]\!\!\right] \text{iff } a' = a$$

$$(\gamma, (a, b), \delta) \in \left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : B, \Delta}{\Gamma, A \otimes B, \Delta}\ (\otimes)\right]\!\!\right] \text{iff } (\gamma, a) \in [\![\pi]\!] \text{ and } (b, \delta) \in [\![\rho]\!]$$

$$\dots$$

$$(\gamma, \delta) \in \left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : A^\perp, \Delta}{\Gamma, \Delta}\ (cut)\right]\!\!\right] \text{iff } \exists a \in |A|, (\gamma, a) \in [\![\pi]\!] \text{ and } (a, \delta) \in [\![\rho]\!]$$

"The" quantitative relational model of linear logic

$$|A \otimes B| = |A| \times |B| \quad |A^{\perp}| = |A| \quad |!A| = \mathfrak{M}_f(|A|) \qquad [\![\pi : \Gamma]\!] \in \mathbf{S}^{|\Gamma|}$$

$$\left[\!\!\left[\frac{}{A^{\perp}, A}\ (ax)\right]\!\!\right]_{a',a} = \delta_{a',a}$$

$$\left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : B, \Delta}{\Gamma, A \otimes B, \Delta}\ (\otimes)\right]\!\!\right]_{\gamma,(a,b),\delta} = [\![\pi]\!]_{\gamma,a} \times [\![\rho]\!]_{b,\delta}$$

$$\dots$$

$$\left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : A^{\perp}, \Delta}{\Gamma, \Delta}\ (cut)\right]\!\!\right]_{\gamma,\delta} = \sum_{a \in |A|} [\![\pi]\!]_{\gamma,a} \times [\![\rho]\!]_{a,\delta}$$

To finiteness spaces. . .

Finiteness spaces (Ehrhard, early 2000's, pub. 2005)

$$\mathcal{F} \subseteq \mathfrak{P}(A) \quad \rightsquigarrow \quad \mathcal{F}^\perp := \{a' \subseteq A;\ \forall a \in \mathcal{F},\ a \perp a'\}$$

$$a \perp a' \qquad \text{iff} \qquad a \cap a' \text{ is finite}$$

- A *finiteness space* is $A = (|A|, \mathcal{F}A)$ with $\mathcal{F}A = (\mathcal{F}A)^{\perp\perp} \subseteq \mathfrak{P}(|A|)$
- The associated vector space: $\mathbf{S}\langle A\rangle := \{x \in \mathbf{S}^{|A|};\ \mathsf{supp}(x) \in \mathcal{F}A\}$

$$\langle x \mid y\rangle := \sum_{a\in|A|} x_a y_a \text{ when } x \in \mathbf{S}\langle A\rangle \text{ and } y \in \mathbf{S}\langle A^\perp\rangle$$

Interpreting LL

$$\mathcal{F}(A^\perp) := (\mathcal{F}A)^\perp$$
$$\mathcal{F}(A \otimes B) := \{x \times y;\, x \in \mathcal{F}A,\, y \in \mathcal{F}B\}^{\perp\perp}$$
$$\mathcal{F}(!A) := \{\mathfrak{M}_f(x);\, x \in \mathcal{F}A\}^{\perp\perp}$$

Theorem (Ehrhard, 2005)

$$[\![\pi : \Gamma]\!]^{Rel} \in \mathcal{F}[\![\Gamma]\!]$$

In particular $[\![\pi : \Gamma, A]\!]^{QRel}_{\gamma,-} \in \mathbf{S}\langle[\![A]\!]\rangle$ and $[\![\rho : A^\perp, \Delta]\!]^{QRel}_{-,\delta} \in \mathbf{S}\langle[\![A]\!]^\perp\rangle$

Interpreting LL

$$\mathcal{F}(A^{\perp}) := (\mathcal{F}A)^{\perp}$$
$$\mathcal{F}(A \otimes B) := \{x \times y;\, x \in \mathcal{F}A,\, y \in \mathcal{F}B\}^{\perp\perp}$$
$$\mathcal{F}(!A) := \{\mathfrak{M}_f(x);\, x \in \mathcal{F}A\}^{\perp\perp}$$

Theorem (Ehrhard, 2005)

$$[\![\pi : \Gamma]\!]^{Rel} \in \mathcal{F}[\![\Gamma]\!]$$

In particular $[\![\pi : \Gamma, A]\!]^{QRel}_{\gamma,-} \in \mathbf{S}\langle[\![A]\!]\rangle$ and $[\![\rho : A^{\perp}, \Delta]\!]^{QRel}_{-,\delta} \in \mathbf{S}\langle[\![A]\!]^{\perp}\rangle$

A semantics of *finite* interaction, e.g.:

- YES: system T;
- NO: letrec, fixpoint operators.

Transport of finiteness structures

Lemma (Ehrhard, 2005)

$$\mathcal{F}(A \otimes B) = \{z \in |A \otimes B|;\ z_A \in \mathcal{F}A,\ z_B \in \mathcal{F}B\}$$
$$\mathcal{F}(!A) = \{\bar{x} \in |!A|;\ \mathsf{supp}(\bar{x}) \in \mathcal{F}A\}$$

Lemma (Transport [T.-V., 2018])

If A is a set and B is a finite space, and $f \subset A \times |B|$ is such that $f \cdot a \in \mathcal{F}B$ for each $a \in A$, then $(A, \{x \subset A;\ f \cdot x \in \mathcal{F}B\})$ is a finiteness space.

Taylor expansion

A long story short:

- finiteness spaces: maps in the coKleisly = power series
- differential λ-calculus (and DiLL)
- Taylor expansion of λ-terms (and of MELL proof nets)

$$\mathcal{T}(-) : \Lambda \to \mathbf{S}^{\Delta}$$

$$\mathcal{T}(MN) := \sum_{n \geq 0} \frac{1}{n!} \mathcal{T}(M) \mathcal{T}(N)^n$$

$$(\lambda x.s)[t_1, \ldots, t_n] \to \sum_{f \in \mathfrak{S}_n} s[t_1/x_1, \ldots, t_n/x_n]$$

Theorem (Ehrhard–Regnier, 2008)

$$\mathsf{NF}(\mathcal{T}(M)) = \mathcal{T}(\mathcal{B}(M))$$

Taylor expansion in a non uniform setting

$$\mathcal{T}(-) : \Lambda_+ \to \mathbf{S}^{\Delta}$$

$$\mathcal{T}(M + N) \coloneqq \mathcal{T}(M) + \mathcal{T}(N)$$

What about:

$$\mathsf{NF}(\mathcal{T}(Y(\lambda x.a + x))) = a + a + a + \ldots ?$$

Taylor expansion in a non uniform setting

$$\mathcal{T}(-) : \Lambda_+ \to \mathbf{S}^\Delta$$
$$\mathcal{T}(M + N) \coloneqq \mathcal{T}(M) + \mathcal{T}(N)$$

What about:

$$\mathsf{NF}(\mathcal{T}(Y(\lambda x.a + x))) = a + a + a + \ldots ?$$
$$\mathsf{NF}(\mathcal{T}(Y(\lambda x.a - x))) = a - a + a - \ldots ?$$

Taylor expansion in a non uniform setting

$$\mathcal{T}(-) : \Lambda_+ \to \mathbf{S}^{\Delta}$$
$$\mathcal{T}(M + N) \coloneqq \mathcal{T}(M) + \mathcal{T}(N)$$

What about:

$$\mathsf{NF}(\mathcal{T}(Y(\lambda x.a + x))) = a + a + a + \ldots ?$$
$$\mathsf{NF}(\mathcal{T}(Y(\lambda x.a - x))) = a - a + a - \ldots ?$$

This never happens for typed terms. Why?

To finiteness spaces and beyond!

Finiteness and Taylor expansion

Let $\mathcal{F}_0 \coloneqq \{\uparrow s_0;\, s_0 \in \Delta_0\}^\perp$ with $s \in \uparrow s_0$ iff $s_0 \in \mathsf{supp}(\mathsf{NF}(s))$.
Define $[\![A]\!] \subseteq \mathcal{F}_0 \subseteq \mathfrak{P}(\Delta)$ by reducibility.

Theorem (Ehrhard, 2010)

If $\vdash M : A$ *then* $\mathcal{T}(M) \in \mathbf{S}\langle [\![A]\!] \rangle$.

Finiteness and Taylor expansion

Let $\mathcal{F}_0 \coloneqq \{\uparrow s_0;\, s_0 \in \Delta_0\}^\perp$ with $s \in \uparrow s_0$ iff $s_0 \in \mathsf{supp}(\mathsf{NF}(s))$.
Define $[\![A]\!] \subseteq \mathcal{F}_0 \subseteq \mathfrak{P}(\Delta)$ by reducibility.

Theorem (Ehrhard, 2010)

If $\vdash M : A$ *then* $\mathcal{T}(M) \in \mathbf{S}\langle [\![A]\!] \rangle$.

Refine: $\mathcal{F} \coloneqq \{\uparrow B;\, B \in \mathcal{B}\}^\perp$ with $B \in \mathcal{B}$ if the size of bags in terms of B is bounded.

Theorem (P.T.V., 2016)

M *is SN iff* $\mathcal{T}(M) \in \mathbf{S}\langle \mathcal{F} \rangle$.

Finiteness and Taylor expansion

Let $\mathcal{F}_0 \coloneqq \{\uparrow s_0;\ s_0 \in \Delta_0\}^\perp$ with $s \in \uparrow s_0$ iff $s_0 \in \mathsf{supp}(\mathsf{NF}(s))$.
Define $[\![A]\!] \subseteq \mathcal{F}_0 \subseteq \mathfrak{P}(\Delta)$ by reducibility.

Theorem (Ehrhard, 2010)

If $\vdash M : A$ *then* $\mathcal{T}(M) \in \mathbf{S}\langle [\![A]\!] \rangle$.

Refine: $\mathcal{F} \coloneqq \{\uparrow B;\ B \in \mathcal{B}\}^\perp$ with $B \in \mathcal{B}$ if the size of bags in terms of B is bounded.

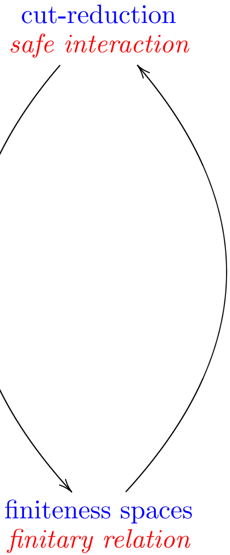
Theorem (P.T.V., 2016)

M *is SN iff* $\mathcal{T}(M) \in \mathbf{S}\langle \mathcal{F} \rangle$.

And then $\mathsf{NF}(\mathcal{T}(M)) = \mathcal{T}(\mathsf{NF}(M))$?

- If $M \to_\beta N$ and $\mathcal{T}(M) \in \mathbf{S}\langle \mathcal{F}_0 \rangle$ then $\mathsf{NF}(\mathcal{T}(M)) = \mathsf{NF}(\mathcal{T}(N))$ [V., 2017]
- This can be adapted to MELL proof nets [Chouquet-V., 2018]

Finiteness and Cut-reduction



Once upon a time there was… MLL

π net

cut-reduction
safe interaction

π cut-free net

π is AC
implies
$\forall \pi'$ AC $\langle \pi | \pi' \rangle$ is WN

Girard
1987

Béchet
1998

$\forall \pi'$ AC $\langle \pi | \pi' \rangle$ is WN
implies
π is AC

MLL
nets
switching acyclicity

π is AC
implies
$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ clique

Girard
1987

Retoré
1994

$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ clique
implies
π is AC

coherence spaces
cliques

and what about exponentials, i.e. MELL?

π net

cut-reduction
safe interaction

π cut-free net

π is AC
implies
$\forall \pi'$ AC $\langle \pi | \pi' \rangle$ is WN

Girard
1987

MELL
nets
switching acyclicity

π is AC
implies
$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ clique

Girard
1987

coherence spaces
cliques


Il faut laisser parler
les modèles !

and what about exponentials, i.e. MELL?

π net

cut-reduction
safe interaction

π cut-free net

π is AC
implies
$\forall \pi'$ AC $\langle \pi | \pi' \rangle$ is WN

Girard
1987

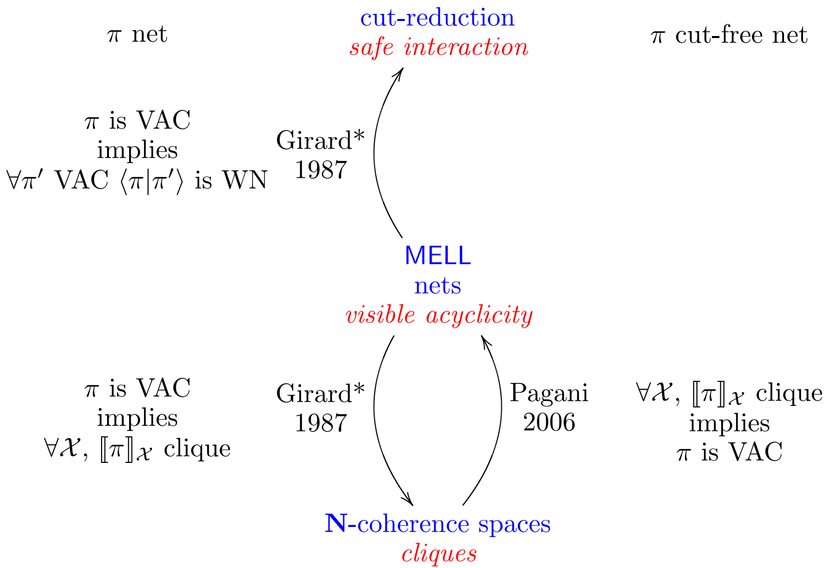
MELL
nets
switching acyclicity

π is AC
implies
$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ clique

Girard
1987

coherence spaces
cliques

and what about exponentials, i.e. MELL? VAC



...what happens in differential LL?

π net

cut-reduction
safe interaction

π cut-free net

π is VAC
implies
$\forall \pi'$ VAC $\langle \pi | \pi' \rangle$ is WN

$\forall \pi'$ VAC $\langle \pi | \pi' \rangle$ is WN
implies
π is VAC

differential nets
with promotion
visible acyclicity

π is VAC
implies
$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ is finitary

$\forall \mathcal{X}$, $[\![\pi]\!]_{\mathcal{X}}$ is finitary
implies
π is VAC

finiteness spaces
finitary relation

The Topological Space Ranger

Linear Logic Cut Rule is given by Linear Algebra composition of Matrices

$$\left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : A^{\perp}, \Delta}{\Gamma, \Delta}\ (cut)\right]\!\!\right]_{\gamma,\delta} = \sum_{a \in |A|} [\![\pi]\!]_{\gamma,a} \times [\![\rho]\!]_{a,\delta}$$

LL Exponentials induce Topological Vector Spaces and Taylor Expansion

A proof of $\mathbf{E} \Rightarrow \mathbf{F} \coloneqq\ !E \multimap F$ is given by $Q \in (\mathbf{K})^{\mathfrak{M}_f(|X|) \times |Y|}$ s.t.

$$\forall x \in \mathbf{S}\langle X\rangle,\ Q(x) = \left(\sum_{m \in \mathfrak{M}_f(|X|)} Q_{m,b} \prod_{a \in m} x_a^{m(a)}\right)_b \in \mathbf{S}\langle Y\rangle$$

Linear Logic Cut Rule is given by Linear Algebra composition of Matrices

$$\left[\!\!\left[\frac{\pi : \Gamma, A \qquad \rho : A^{\perp}, \Delta}{\Gamma, \Delta}\ (cut)\right]\!\!\right]_{\gamma,\delta} = \sum_{a \in |A|} [\![\pi]\!]_{\gamma,a} \times [\![\rho]\!]_{a,\delta}$$

LL Exponentials induce Topological Vector Spaces and Taylor Expansion

A proof of $\mathbf{E} \Rightarrow \mathbf{F} := !E \multimap F$ is given by $Q \in (\mathbf{K})^{\mathfrak{M}_f(|X|) \times |Y|}$ s.t.

$$\forall x \in \mathbf{S}\langle X\rangle,\ Q(x) = \left(\sum_{m \in \mathfrak{M}_f(|X|)} Q_{m,b} \prod_{a \in m} x_a^{m(a)}\right)_b \in \mathbf{S}\langle Y\rangle$$

Topology gives a meaning to **converging sums**.

Topological Vector Spaces based on **Sequences**

Köthe Spaces [Ehrhard 2003]

- A set $|A|$ induces a real or complex vector space $\mathbf{K}^{|A|}$
- An orthogonality between vectors

$$x \perp x' \qquad \text{iff} \qquad \sum_{a \in |A|} |x_a \, x_{a'}| \text{ is absolutely converging}$$

- If $E \subset \mathbf{K}^{|A|}$; $E = E^{\perp\perp}$, then it is equipped with a locally convex topology

Topological Vector Spaces based on Sequences

Köthe Spaces [Ehrhard 2003]

- A set $|A|$ induces a real or complex vector space $\mathbf{K}^{|A|}$
- An orthogonality between vectors

$$x \perp x' \qquad \text{iff} \qquad \textstyle\sum_{a \in |A|} |x_a \, x_{a'}| \text{ is absolutely converging}$$

- If $E \subset \mathbf{K}^{|A|}$; $E = E^{\perp\perp}$, then it is equipped with a locally convex topology

Finiteness spaces [Ehrhard 2002]

- A set $|A|$ and a space of sequences $\mathbf{S}\langle A \rangle := \{x \in \mathbf{S}^{|A|};\, \mathsf{supp}(x) \in \mathcal{F}A\}$
- An orthogonality between vectors

$$x \perp x' \qquad \text{iff} \qquad \textstyle\sum_{a \in |A|} x_a \, x_{a'} \text{ is finite}$$

- A linearized topology induced by linear neighbourhoods

$$\mathcal{V}(v) = \{x \in \mathbf{S}\langle A \rangle;\, \mathsf{supp}(x) \cap v = \emptyset\},\ v \in \mathcal{F}A^{\perp}$$

- A *dual* linearized bornology induced by compact/bounded subspaces

$$\mathcal{K}(u) = \{x \in \mathbf{S}\langle A \rangle;\, \mathsf{supp}(x) \subset u\},\ u \in \mathcal{F}A$$

Topological Vector Spaces NOT based on Sequences

Lefschetz spaces [T09]

- A linearized Topological Vector Spaces
- An orthogonality between opens U and bounded linear subspaces B

  $$U \perp B \qquad \text{iff} \qquad B/B \cap U \text{ has finite dimension}$$

  In a bornological vector space is a Lefschetz space such that $(\mathcal{V})^{\perp\perp} = \mathcal{V}$
- Open Problem: prove the stability of bornological Lefschetz spaces by tensor product and exponentials

Topological Vector Spaces **NOT** based on **Sequences**

Lefschetz spaces [T09]

- A linearized Topological Vector Spaces
- An orthogonality between opens U and bounded linear subspaces B

$$U \perp B \qquad \text{iff} \qquad B/B \cap U \text{ has finite dimension}$$

  In a bornological vector space is a Lefschetz space such that $(\mathcal{V})^{\perp\perp} = \mathcal{V}$

- Open Problem: prove the stability of bornological Lefschetz spaces by tensor product and exponentials

Convenient Vector Spaces

- Mackey-complete, separated, topological convex bornological vector spaces and bounded linear maps
- Smooth maps preserve smooth curves (no need of norm) [BET 10]
- Maps that are Series preserve holomorphic curves [KT12]
- Schwartz ε-Tensor and reflexive models [Dabrowski & Kerjean 17]

Probabilistic setting

Probabilistic Coherent Spaces are based on Sequences

- A set $|A|$ and a cone of sequences $\mathbf{R}^{+|A|}$
- An orthogonality between vectors

$$x \perp x' \qquad \text{iff} \qquad \sum_{a \in |A|} |x_a \, x_{a'}| < 1$$

- Morphims are Taylor series
- Topology induces Full Abstraction [EPT14]

Probabilistic setting

Probabilistic Coherent Spaces are based on Sequences

- A set $|A|$ and a cone of sequences $\mathbf{R}^{+|A|}$
- An orthogonality between vectors

$$x \perp x' \qquad \text{iff} \qquad \sum_{a \in |A|} |x_a \, x_{a'}| < 1$$

- Morphims are Taylor series
- Topology induces Full Abstraction [EPT14]

Probabilistic Stable Cones are NOT based on Sequences

- Cones and ∞-non-decreasing functions
- Measurable maps preserve measurable paths (no need of measurability)
- A model of probabilistic PCF [EPT18]
- A model of Linear Logic [Ehrhard20]
- Topology induces Stable morphims between Probabilistic Coherent Spaces are series ! [Crubille19]

A message by
Martin Hyland