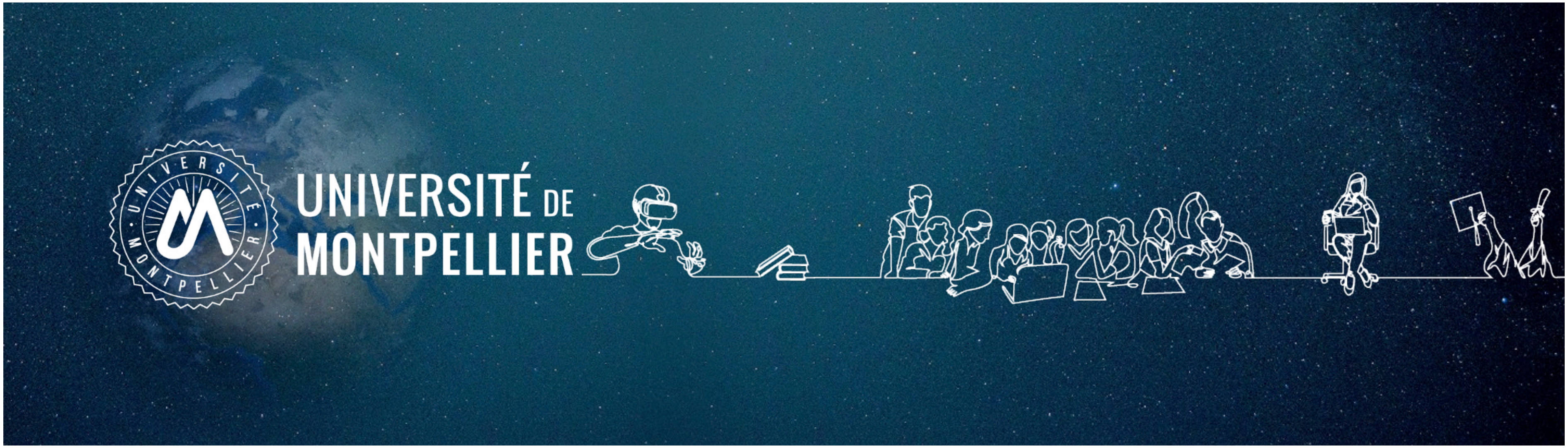



# PROOF NETS AND COHERENCE SPACES (oldies)

Talk @ **THOMAS EHRHARD 60** Festschrift

CHRISTIAN RETORÉ — LIRMM & UNIVERSITÉ DE MONTPELLIER

# HISTORICAL REMARKS

- Old stuff for the birthday
  of an old friend
  — who still is not that old!

- Quite an easy talk
  just a warm up with low level reminders
  before serious talks take place

- Thomas from 1986 – 1987 DEA Girard / Krivine

- Friends and neighbours with Pasquale Malacaria
  around rue mouffetard 1991

- Visits Marseilles / Nice in the early 90s
  (highly coherent period)

- Then workshop, committees etc.

# Trade unionist remarks

- About this 29 september 2022 in France
- Secondary school teachers, school teachers are on strike
- They have been loosing a lot of purchasing power in the last 20 years
- Worse their working conditions have deteriorated considerably since the 80s
- Not enough persons want to become teachers anymore.

**Without teachers we would not be academics today**

# Coherence spaces (Girard 1986)

A short reminder – That's where linear logic took place, well worth a visit!

# Denotational semantics, categorical interpretation

Proof π of C under asumption H

Morphism from [[H]] to [[C]]

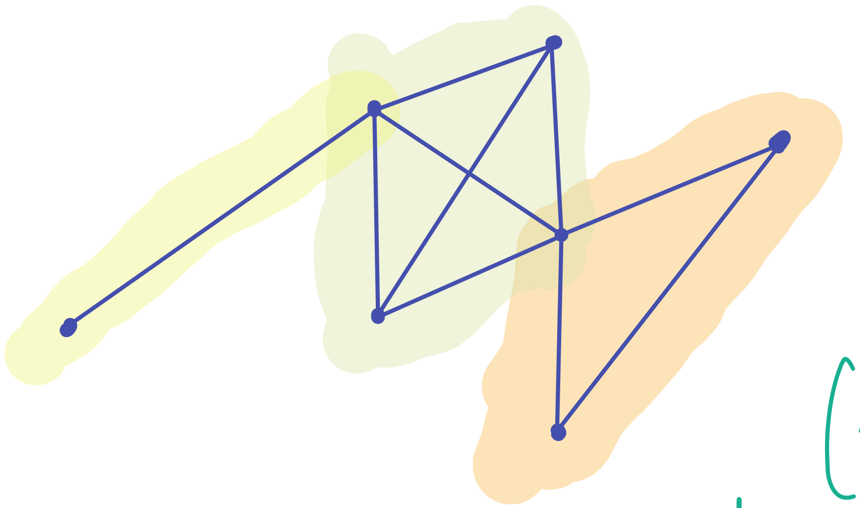
When π reduces to π' … π unchanged i.e. [[π]]=[[π']]

(full abstraction, denotational completeness....
Sequentiality ;-)

# Coherence space A

**Web : (countable set of tokens) |A|**

**A binary irreflexive relation on it ⌢ (simple graph)**



objects: cliques (not all of them) maximal, total ...

Linear Negation: complement graph

$a \frown b\,[A]$ iff $a \smile b\,[A^\perp]$



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## Stable maps

stable $\beta \in F(a)$

$\exists\, a_0 \subset A$ min $\beta \in F(a_0)$

$F : (a_0, \beta)$

**Approximants**

**Representation of Hom(A,B) as a coherence space.**

$F$ : cliques to cliques

- $a \subset b \quad F(a) \subset F(b)$
- $F(\bigcup a_i) = \bigcup (F(a_i))$ $\quad \bigcup$ : directed union
- if $a \cup b$ clique <u>then</u> $F(a \cap b) = F(a) \cap F(b)$

with & product $\qquad C\ C\ C$



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## Linear maps

Coherence spaces with different morphisms.

$\exists \alpha \in A \; \beta \in F(\alpha)$ $\beta \in F(x)$ $\beta \in F(\downarrow \alpha)$ $F : (\alpha, \beta)$

linear maps instead of stable maps

Not a CCC $\otimes$ is not a product.

$F(\bigcup a_i) = \bigcup F(a_i)$ when all $a_i\ a_j$ are pairwise compatible

(or F definable on tokens)

## Linear connectives

$|A * B| = |A| \times |B|$

just the two of them (when commutative)

| $\otimes$ | $\smile$ | $=$ | $\frown$ |
|---|---|---|---|
| $\smile$ | $\smile$ | $\smile$ | $\smile$ |
| $=$ | $\smile$ | $=$ | $\frown$ |
| $\frown$ | $\smile$ | $\frown$ | $\frown$ |

| $⅋$ | $\smile$ | $=$ | $\frown$ |
|---|---|---|---|
| $\smile$ | $\smile$ | $\smile$ | $\frown$ |
| $=$ | $\smile$ | $=$ | $\frown$ |
| $\frown$ | $\frown$ | $\frown$ | $\frown$ |

$|!A| =$ finite cliques of $|A|$ $\quad a \frown a' \; [!A]$ iff $a \cup b$ clique of $A$

$A \multimap B = A^\perp ⅋ B \qquad F(a) = \{\beta \,/\, \alpha \in a \text{ et } (\alpha, \beta) \in f\}$

$A \Rightarrow B = (!A) \multimap B$

# (Multiplicative) Linear Logic (Girard 1987)

Another short reminder

## Axioms, Rules and Cuts (one sided multiplicative)

$$\vdash a, a^\perp \quad (a \vee \neg a)$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\otimes$$

contexts are concatenated

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A ⅋ B, \Delta}⅋$$

Cut

$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

(kind of $\otimes$ $\exists x \; x \otimes x^\perp$)

+ exchange (or not??)



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## Remarks

Extremely simple calculus

→ very elegant, with many properties

→ not very expressive

# The MIX rule

**"and" implies "or" in MLL**

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ \text{mix}$$

$A \otimes B \multimap A ⅋ B$ validated by coherence spaces.

(coherent wrt $\otimes$ implies coherent wrt ⅋)

with units ... complicated.

# Multiplicative Proof nets (Girard 1987)

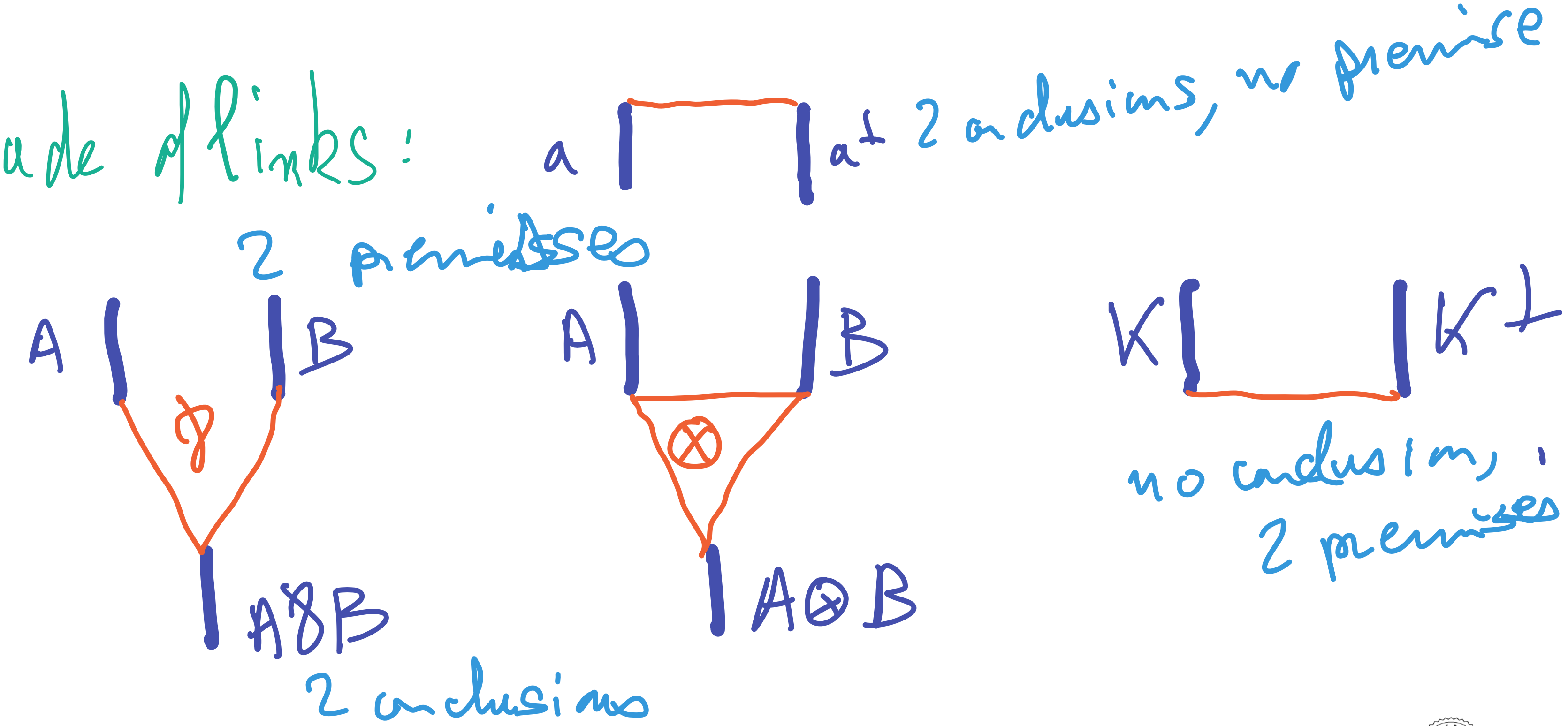
Yet another short reminder

# Graphs denoting proofs

**Up to rule permutations**

**Cut-elimination as graph rewriting**

made of links:



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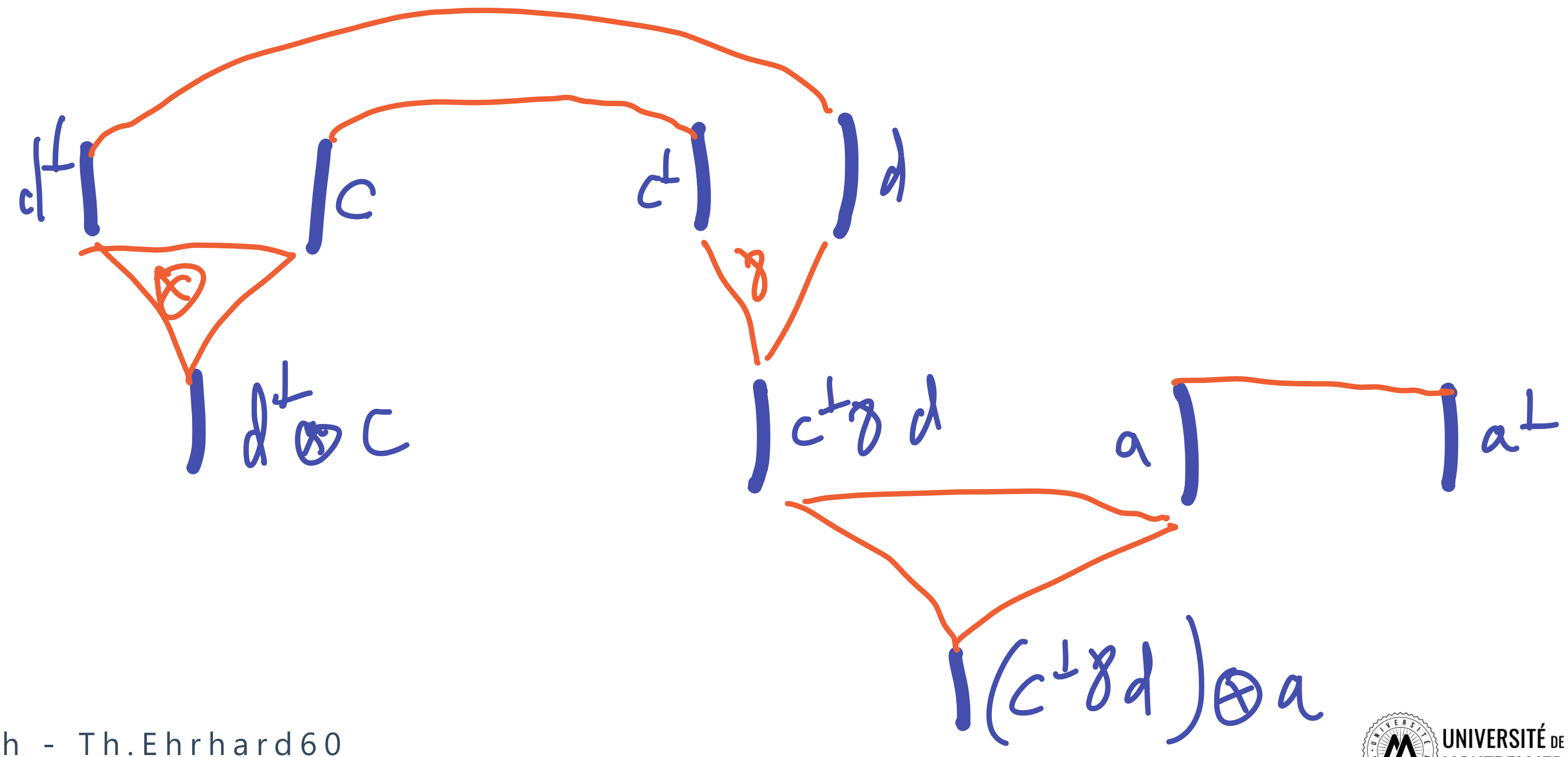
**A natural pile-up of links:**



# Proof nets : criterion à la Danos-Reigner

General

Every alternate elementar cycle contains a chord

Here (because of the shape of links)

No alternate elementary cycle

($\nexists$ an alternate elementary path
betwen any two premises
in order to exclude the Mix rule)



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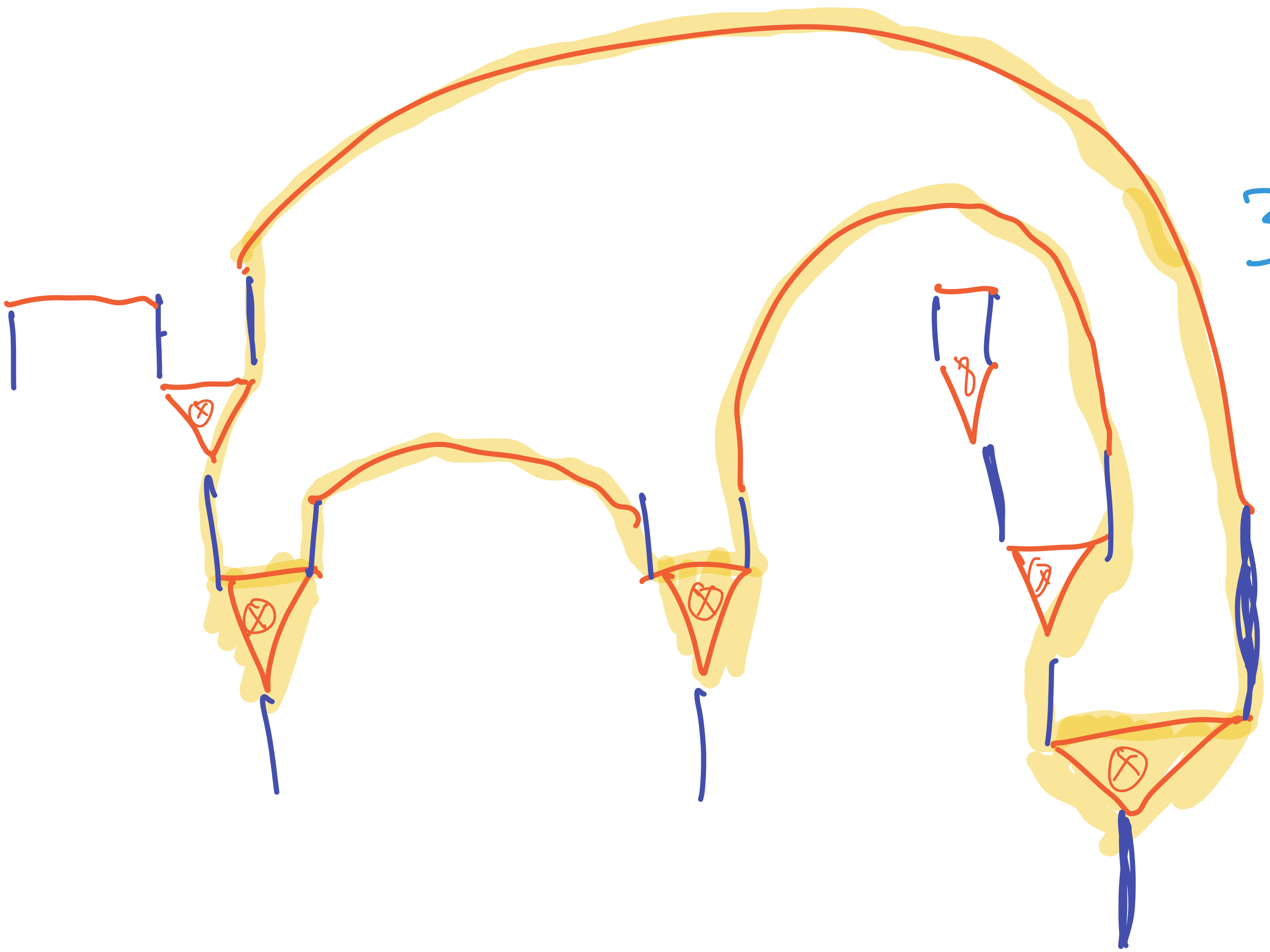
# Proof structures and nets

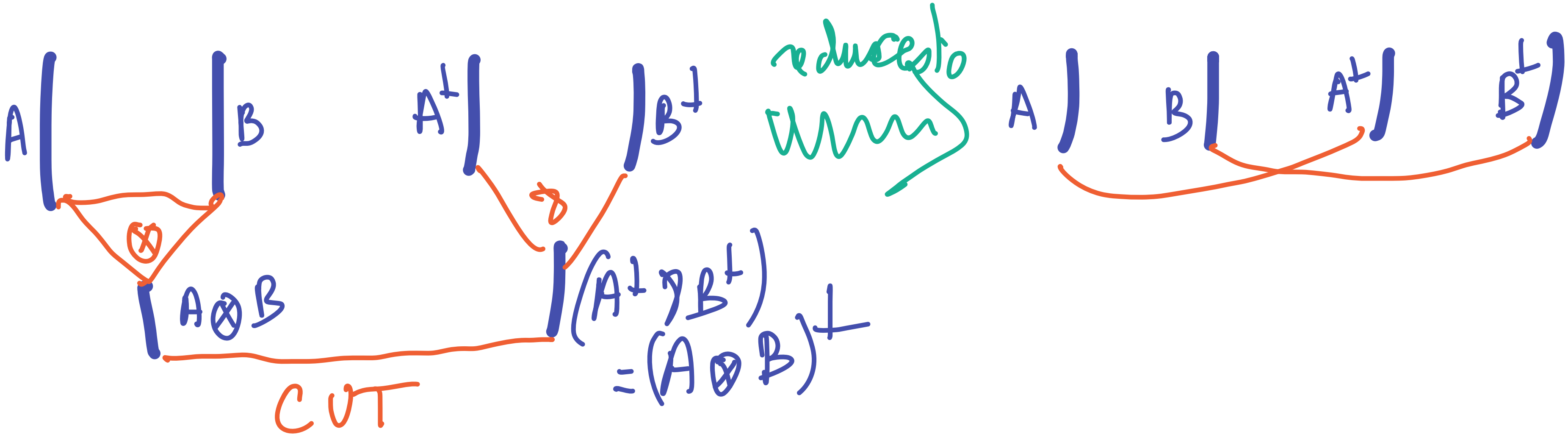


If one of the 3 $\otimes$ in the cycle becomes $\invamp$, proof NET

# Cut-elimination

In MLL(+mix) that's clearly a terminating process

preserves the absence of ae-cycle



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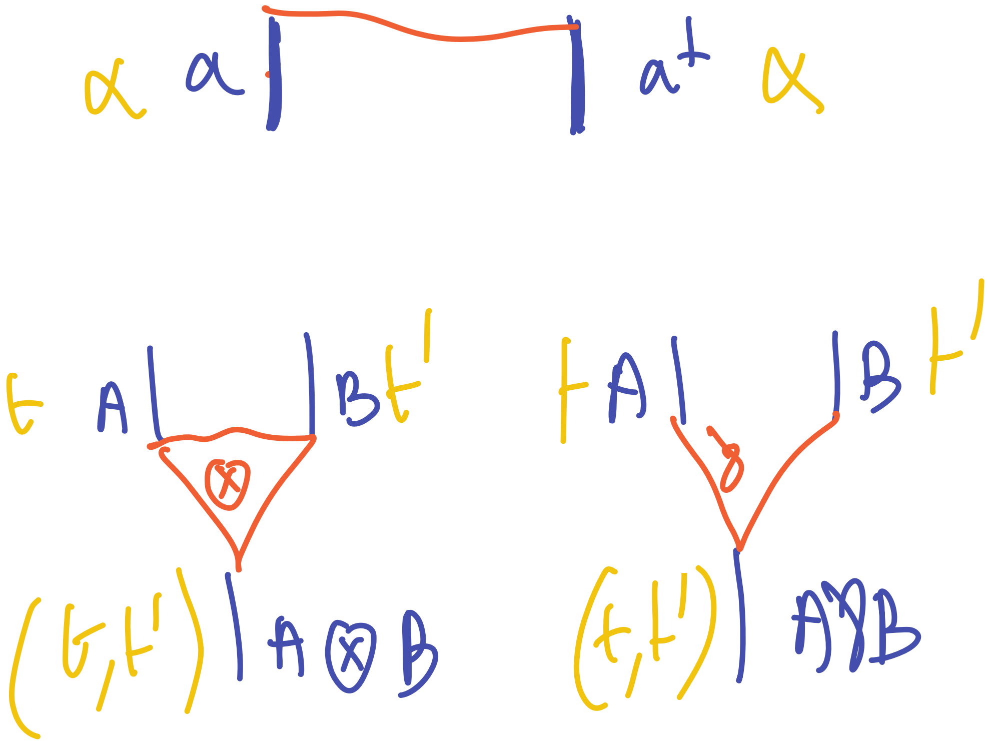
# Interpreting a proof (net) of X as clique of the corresponding coherence space

Experiments (Girard 1987 LL paper) slightly revisited

# Proof structures -> interpretation

## EXPERIMENTS À LA GIRARD BUT UP-SIDE DOWN



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# CUTS?

$$\langle x_1 \; x_n \underbrace{\,|\, K \qquad K^{\perp} \,|\,}\; y_1 \; y_n \rangle$$

$$\text{ax} \mid a \qquad x_1 = y_1 \quad y_2 = y_2 \quad \ldots \quad x_n = y_n$$

## ~~[illegible]~~ Result of an experiment

The tuples collected on conclusions for the succeeding experiments

All the results of all the succeeding experiments in $\pi$ are $[\![\pi]\!]$

1) a clique of $C_1, C_2, \ldots, C_p$

2) preserved by cut elimination

# « Soundness »
(Girard 1987)

# Remarks

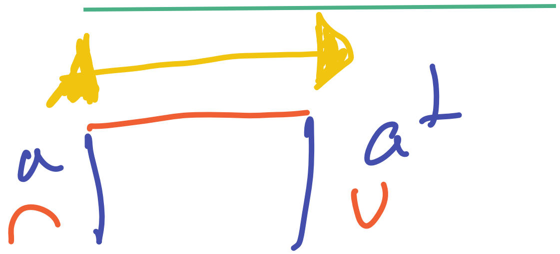
If two experiments differ ~~on~~ some where then they are coherent on some conclusion.

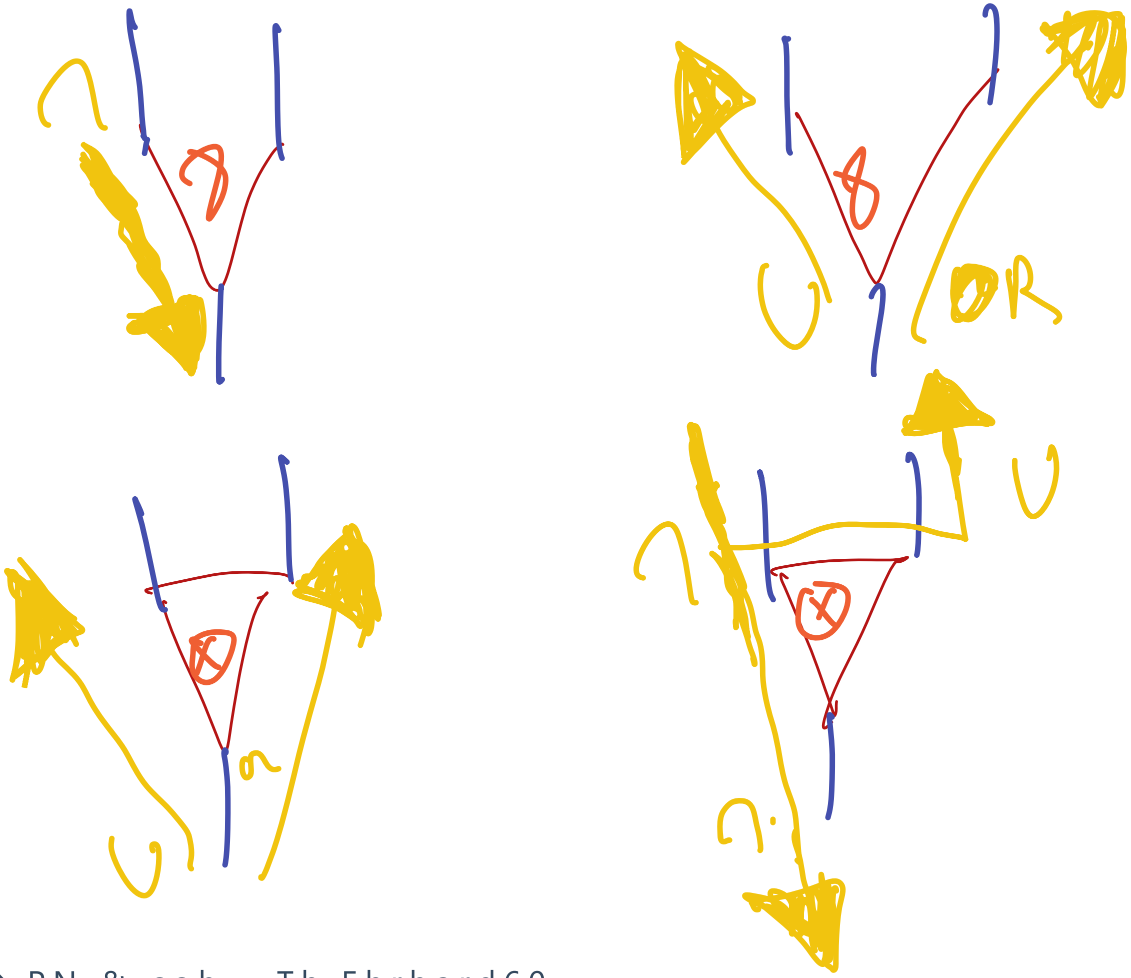
IDEA: extending a path up incoherent $\smile$ down $\frown$ coherent

## Remarks

≠ incoherent ↑ up

≠ coherent ↓ down





# « Completeness »

(Inria report Retoré 1994 paper in MSCS 1997 )
presented in 1995 for Dana Scott honorary degree in Darmstadt

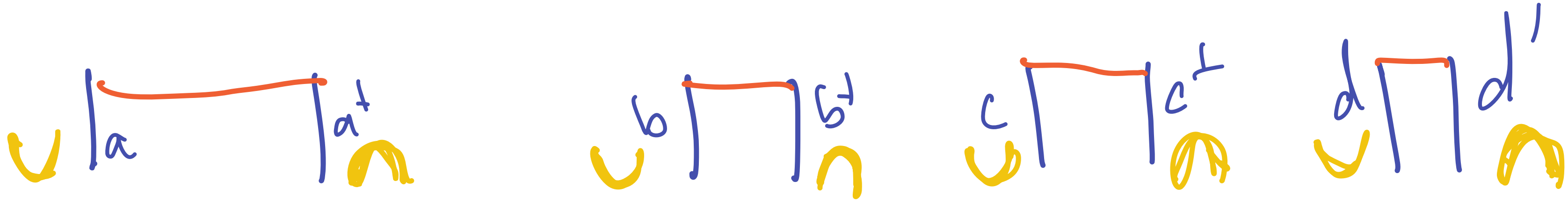
# A remark

We need a coherence space N such that there are

- two coherent tokens in N
- Two incoherent tokens in N (coherent in $N^{\perp}$)

- We take N to be isomorphic to his negation

**Given an ordered sequence of axioms**
**(with a first and a second conclusion)**
**There exist two diferent experiments such that**



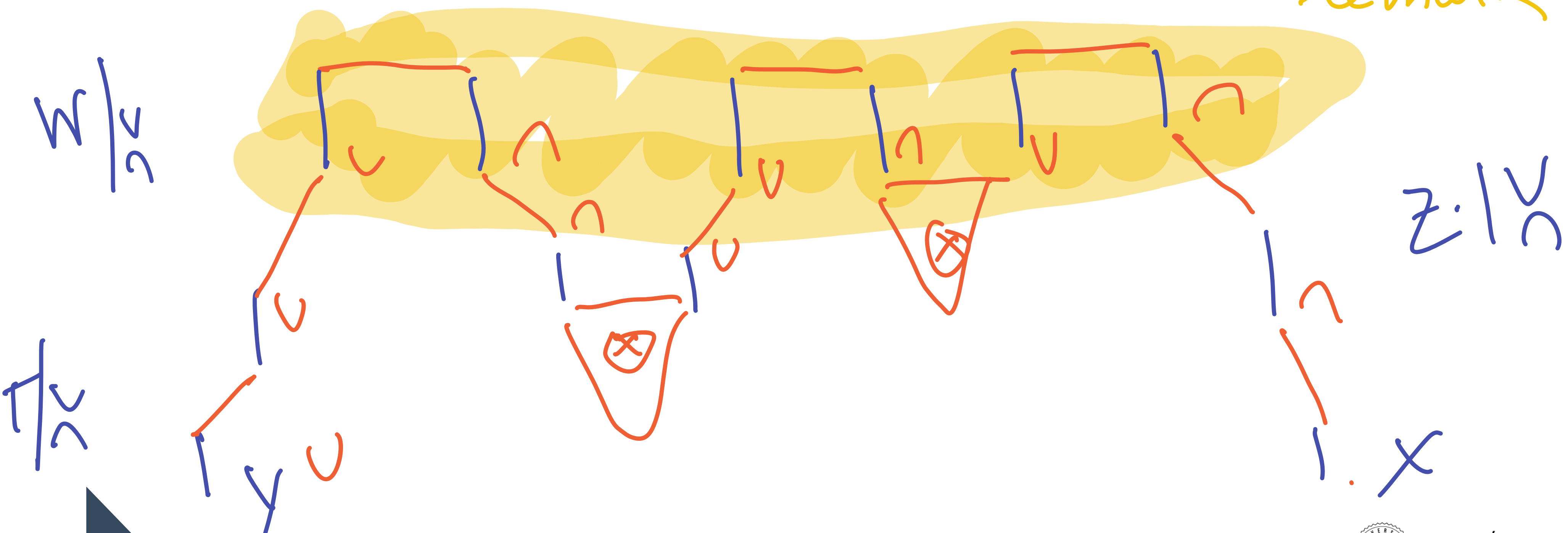

obvious: take a token that is $\frown$ ~~oa~~ $\smile$
with the one you already have

## Lemma

Induction on the proofnet using sequentialisation

In a proof NET when there exist an ae path from a conclusion X to a conclusion Y one may found two experiments such that they are coherent on X, incoherent on Y. and $\overset{\vee}{\frown}$ on any other conclusion

as in the remark



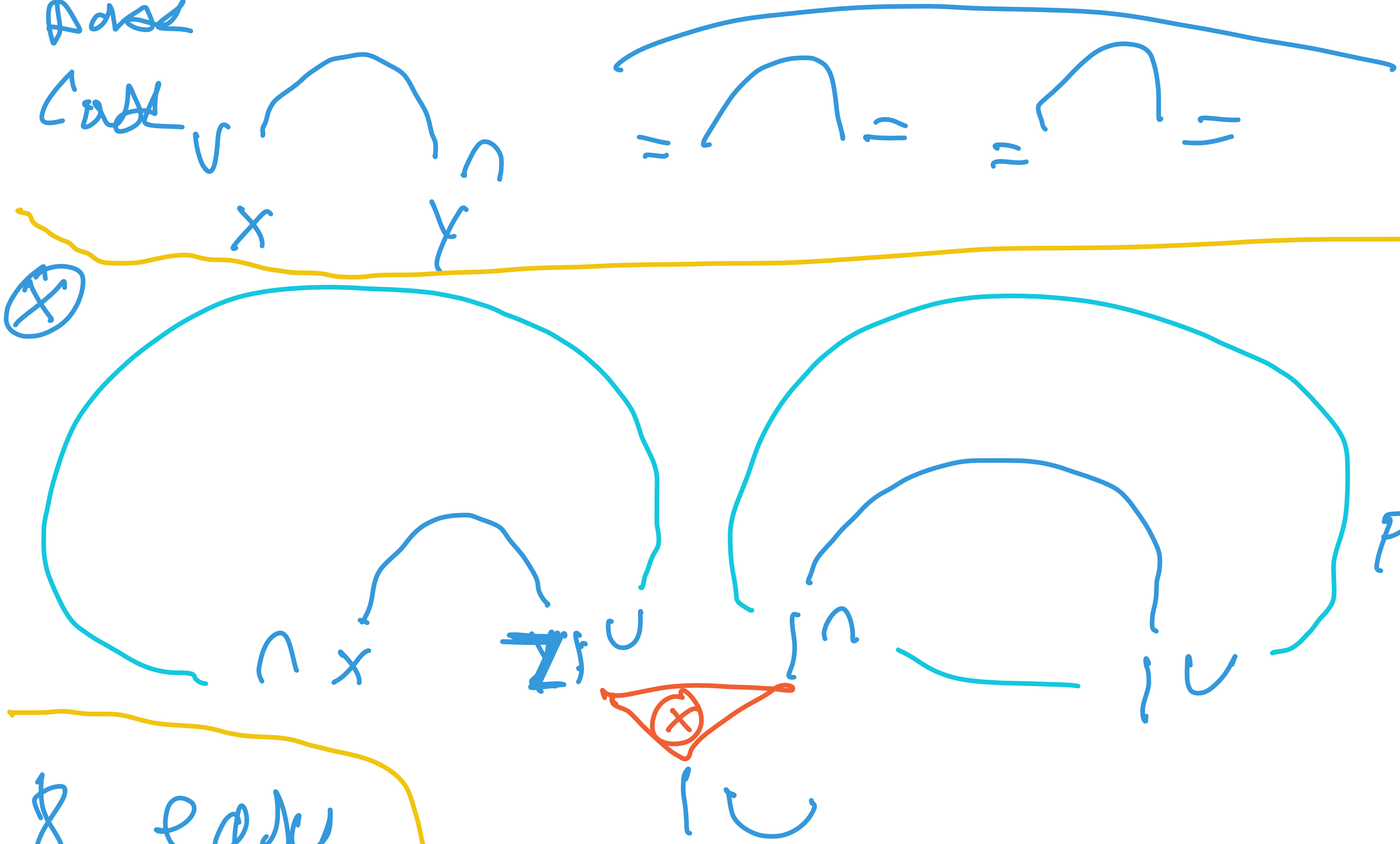
# Completeness

**When a proof structure is not correct it is possible to find two expériment such that the results are incoherent on all conclusion na strictly incoherent on one conclusion — hence incoherent w.r.t. the par of all the conclusions.**

## Completeness / proof

base case

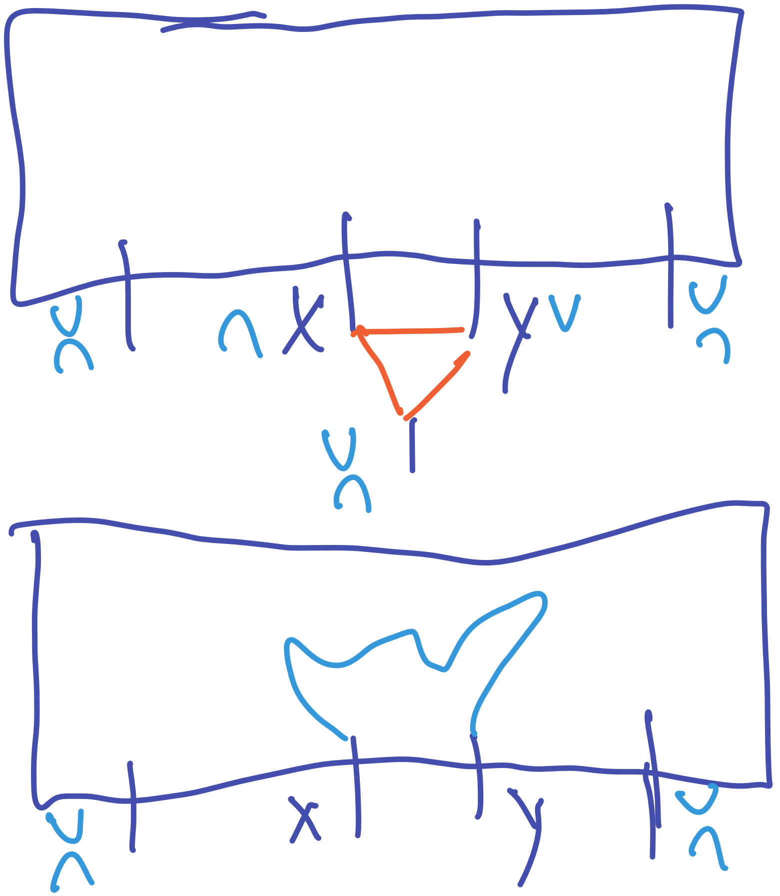
others




path on one side ok

⅋ easy

# Completeness / proof



incorrect ? yes ok

no

Remain X Y



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# Deadlock freeness of the reduct

(Retoré 1994)

# Deadlock free-ness

(Asperti terminology i think)

No loop in the reduced proof STRUCTURE.

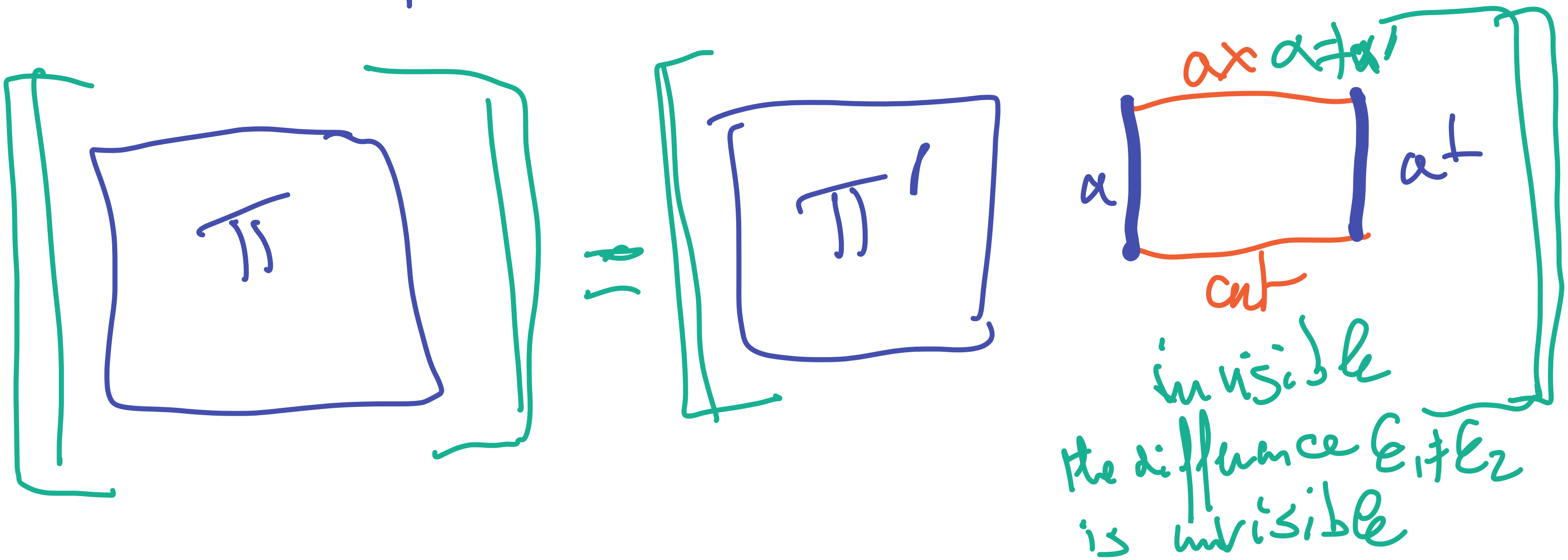
An incorrect

~~A~~ proof structure may well reduce to a proof net.

# Characterisation

Two different experiments yield the same result:
there will be a loop in the normal form.

The interpretation is preserved by cut-elimination



# Extensions to MELL and MALL

Pagani Tasson

# Not easy to extend, because of mix

**While the criterion for boxes is:**
**1) Check whether the inside of a boc is correct, and**
**2) Replace the box with a kind of n-ary axioms yielding the conclusions of the box, is the out side correct?**



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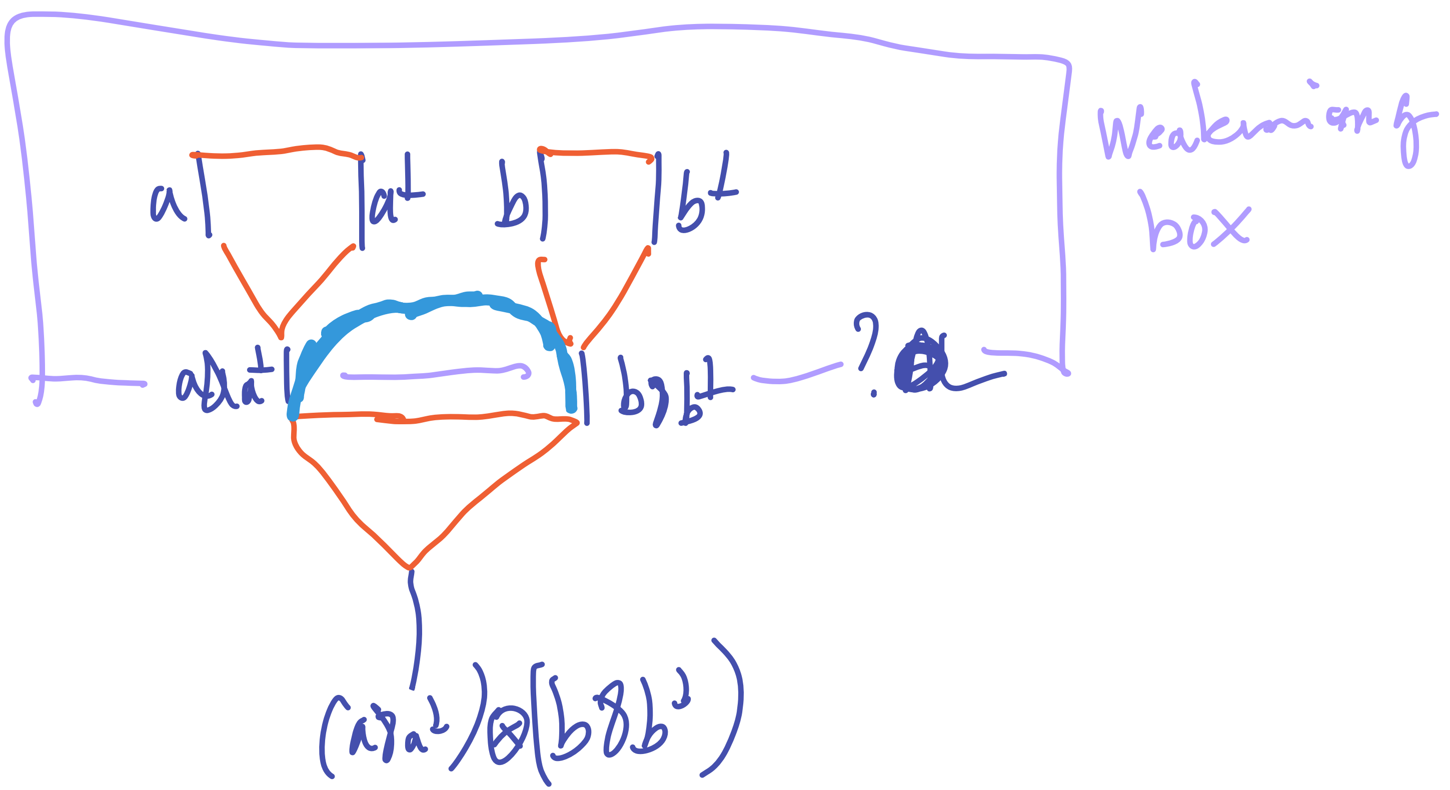
# Example by Michele Pagani

**Michele will explain that tomorrow.** (much better than I can)



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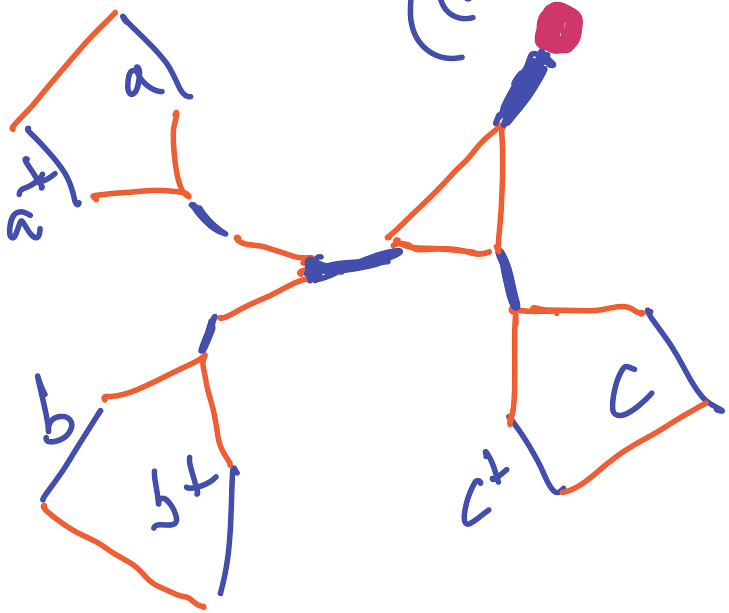
# *Conclusion ⊗ perspectives*

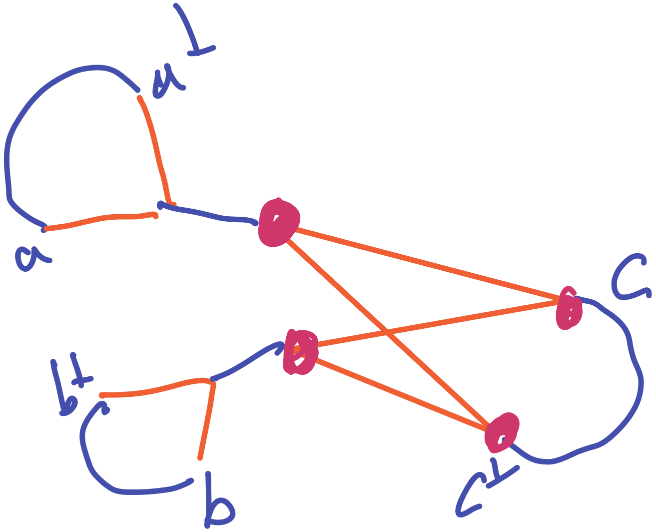
# From RnB graphs to handsome proof nets

**FOLD / UNFOLD**

$$((a \parr a^{\perp}) \parr (b \parr b^{\perp})) \otimes (c \parr c^{\perp})$$



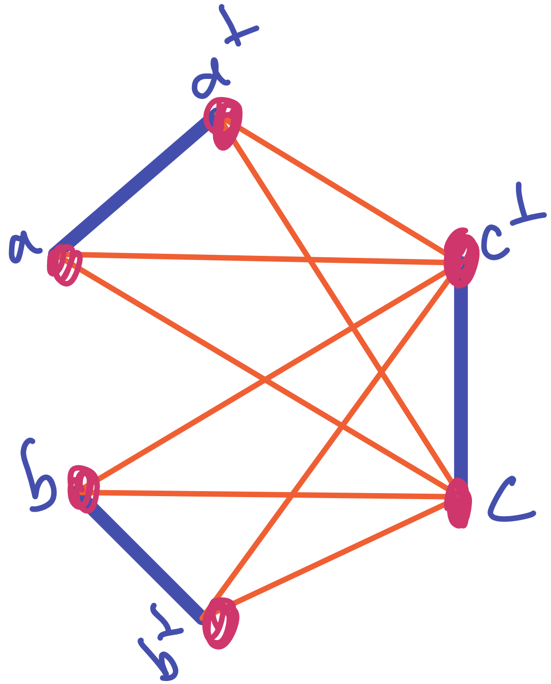

$\rightsquigarrow$



# From RnB graphs to handsome proof nets

**COGRAPH (FORMULA) + PERFECT MATCHING (AXIOMS)**

$$(c \parr c^\perp) \otimes \left( (a \parr a^\perp) \parr (b \parr b^\perp) \right)$$



# Criterion for handsome proof nets Retoré & Ehrhard

AXIOMS : PERFECT MATCHING

CONCLUSION: A COGRAPH DESCRIBING THE FORMULA

THERE IS A CHORD ON EVERY ALTERNATE ELEMENTARY CYCLE

From Thomas' web page:

**Thomas Ehrhard.** *A new correctness criterion for MLL proof nets.* 2014. Accepted at LICS'14. This criterion was first published by C. Rétoré in TCS 294(3):473-488, 2003. I rediscovered it independently (my presentation is slightly more general) and I am convinced that it is worth being further studied. pdf.

unusual! THANKS!



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## Question (discussed with Thomas)

Intuition

- The semantic criterion with coherence spaces
- The handsome criterion

should be related / we know they are equivalent ...

So far no argument, just an intuition.

could we find out why?



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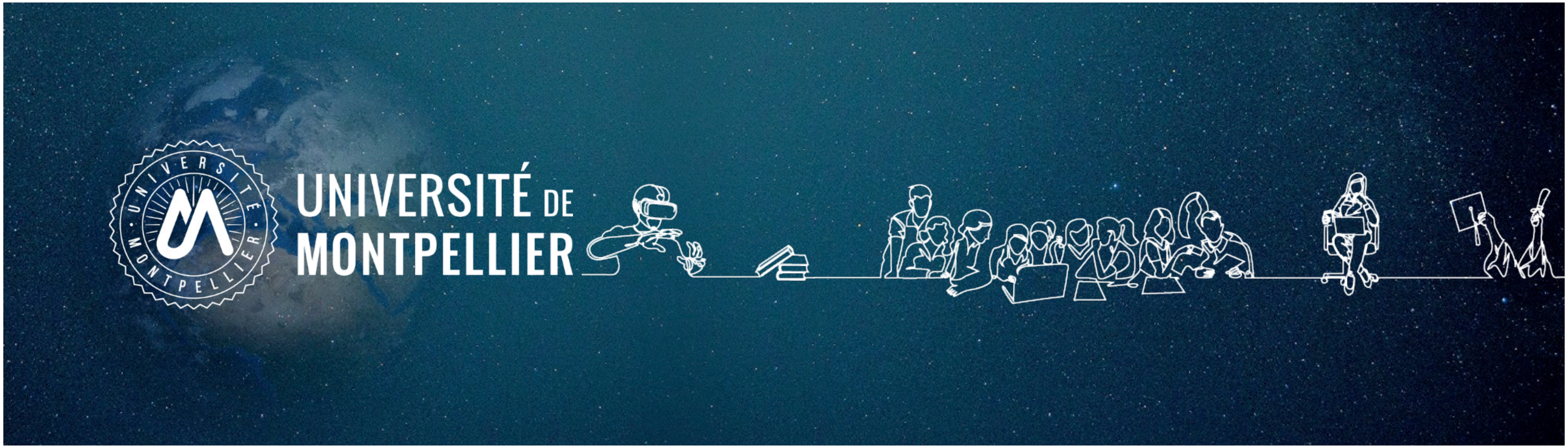
Happy birthday Thomas !




Happy birthday Thomas