Talking with Thomas

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Thomas' 60th birthday
CNAM, Paris
29-30 Sept 2022

Thomas' question:

What is the precise relation between differential nets and Linear Logic experiments (type derivations for LL)?

Linear Logic and Differential Linear Logic: a methodological point

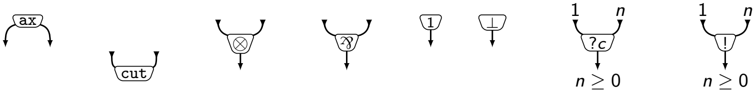
- the methodological approach behind the introduction of Linear Logic and Differential Linear logic is very similar
- denotational semantics gives mathematical counterparts to programming languages: in proof-theory this is the study of the mathematical invariants of the cut-elimination process
- a nice model can reveal some hidden structure of proofs and can suggest improvements of the proof system (and give new insights on Logic)
- Girard's coherent model of the typed λ-calculus: introduction of the exponential connectives and thus of LL proof-nets (a great novelty carried by LL)
- Ehrhard's finiteness spaces: introduction (by Ehrhard-Regnier) of the co-structural rules and the representation of proofs as (possibly infinite) sums of differential nets, which have both a geometric nature (as graphs) and an algebraic one (as elements of the interpretation of proofs).

Taylor expansion of a MELL proof-structure

MELL (DiLL and DiLL_0) formulas:

$$A ::= X \mid A \otimes A \mid A \parr A \mid \bot \mid 1 \mid !A \mid ?A$$

DiLL_0-nodes:



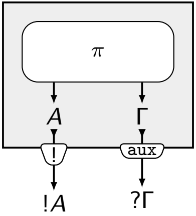
A $?c$-node has $n \geq 0$ premises of type A and one conclusion of type $?A$
A !-node has $n \geq 0$ premises of type A and one conclusion of type $!A$.

DiLL-nodes=DiLL_0-nodes+boxes.
MELL-nodes=DiLL_0-nodes where !-nodes have arity 1 +boxes.

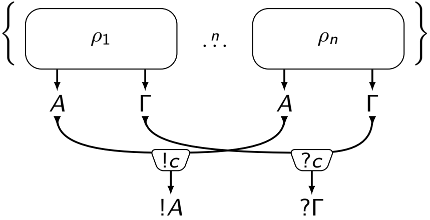
$$\text{(Qualitative) Taylor expansion } \mathcal{T}: \begin{array}{ccc} \mathrm{MELL} & \to & \mathcal{P}(\mathrm{DiLL}_0) \\ \pi & \mapsto & \mathcal{T}(\pi) \end{array}$$

Taylor expansion of a MELL proof-structure: example

Idea: each box is replaced by n copies of its content, recursively (for every box and every $n \in \mathbb{N}$)



$$\stackrel{\mathcal{T}}{\longmapsto} \bigcup_{\substack{n\in\mathbb{N}\\ \rho_1,\dots,\rho_n\in\mathcal{T}(\pi)}}$$



An element of the Taylor expansion of the proof-structure π is itself a (resource) proof-structure and an approximation of π.

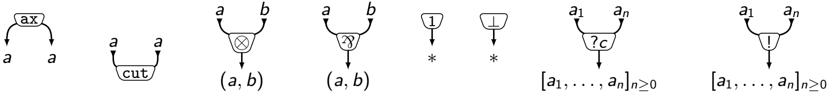
Experiments and interpretation of a DiLL_0 proof-structure

We fix an infinite set $\mathcal{A}t$ of atoms.

Let $|\cdot|$ be the function associating with any MELL formula A the set $|A|$ defined by induction on A as follows:

$$|X| = |X^\perp| = \mathcal{A}t, \text{ for any variable } X; \qquad |1| = |\bot| = \{*\};$$
$$|A \otimes B| = |A \parr B| = |A| \times |B|; \qquad |!A| = |?A| = \mathscr{M}_{\mathrm{fin}}(|A|).$$

An experiment of a DiLL_0 proof-structure ρ is a function (= labelling) e s.t. $p \mapsto e(p) \in |A|$ for any edge $p : A$ of ρ.



The relational interpretation of a DiLL_0 proof-structure ρ with conclusions $p_1 : A_1, \ldots, p_n : A_n$ is $[\![\rho]\!] = \{|e| : e \text{ is an experiment of } \rho\}$, where $|e| = (e(p_1), \ldots, e(p_n))$ is the result of e.

Relational interpretation of a MELL proof-structure

For a proof-structure π, define
$\llbracket \pi \rrbracket = \bigcup_{\rho \in \mathcal{T}(\pi)} \llbracket \rho \rrbracket = \{|e| : e \text{ experiment of } \pi\})$.

"Most informative" points of the interpretation: $a \in |A|$ is injective if every atom occurring in a occurs exactly twice. If $X \subseteq |A|$, we set $X_{inj} = \{a \in X \mid a \text{ is injective}\}$.

The injective interpretation of π (in MELL) and ρ (in DiLL_0) are $\llbracket \pi \rrbracket_{\mathrm{inj}} = \llbracket \pi \rrbracket \cap | \parr \Gamma |_{\mathrm{inj}}$ and $\llbracket \rho \rrbracket_{\mathrm{inj}} = \llbracket \rho \rrbracket \cap | \parr \Gamma |_{\mathrm{inj}}$.

We have: $\llbracket \pi \rrbracket_{\mathrm{inj}} = \bigcup_{\rho \in \mathcal{T}(\pi)} \llbracket \rho \rrbracket_{\mathrm{inj}}$.
(Most informative points: from $\llbracket \pi \rrbracket_{\mathrm{inj}}$ one immediately recovers $\llbracket \pi \rrbracket$).

There are many equivalent (up to renaming) injective points: $a \sim_A a'$ iff there exists a bijection $\sigma : \mathcal{A}t \to \mathcal{A}t$ such that $a = \sigma_A(a')$.

Taylor expansion: a bridge between syntax and semantics

For π normal ($=$ cut-free, η-expanded) MELL proof-structure (or simply typed λ-term), $\rho \in \mathcal{T}(\pi)$ is a canonical representative of an equivalence class of most informative points of $[\![\pi]\!]$: $[\![\pi]\!]_{\mathrm{inj}} / \sim_{\mathcal{?}\Gamma}$ is precisely $\mathcal{T}(\pi)$.

Proposition (Guerrieri-Pellissier-TdF, but also "folklore")

For π normal with conclusion Γ, the quotient of the identity

$$[\![\pi]\!]_{\mathrm{inj}} = \bigcup_{\rho \in \mathcal{T}(\pi)} [\![\rho]\!]_{\mathrm{inj}}$$

through the equivalence $\sim_{\mathcal{?}\Gamma}$ yields a bijection

$$f : \mathcal{T}(\pi) \to [\![\pi]\!]_{\mathrm{inj}} / \sim_{\mathcal{?}\Gamma}$$
$$\rho \mapsto [\![\rho]\!]_{\mathrm{inj}}$$

Remark: If $\pi \to \pi'$ then $\mathcal{T}(\pi) \to^{+} \mathcal{T}(\pi')$ ($\rightsquigarrow$ $\mathcal{T}$ is not invariant under reduction). The semantic meaning of $\mathcal{T}(\pi)$ when π is with cuts is unclear!

For a normal MELL proof-structure (or λ-term) π, we can deal with the elements of $\mathcal{T}(\pi)$ instead of the elements of $[\![\pi]\!]$ $\rightsquigarrow$ a geometrical representation of the relational interpretation of π.

Taylor expansion: a bridge between syntax and semantics (2)

Proof of $[\![\pi]\!]_{\mathrm{inj}/\sim_{\parr\Gamma}} \simeq \mathcal{T}(\pi)$:

FACT 1: ρ DiLL_0 proof-structure with conclusion Γ.
(i) if $x, x' \in [\![\rho]\!]_{\mathrm{inj}}$, then $x \sim_{\parr\Gamma} x'$.
(ii) If $x \in [\![\rho]\!]_{\mathrm{inj}}$, $x' \in |\parr\Gamma|_{\mathrm{inj}}$ and $x \sim_{\parr\Gamma} x'$, then $x' \in [\![\rho]\!]_{\mathrm{inj}}$.

FACT 2: For ρ, ρ' cut-free η-expanded DiLL_0 proof-structures with conclusion Γ, we have that $[\![\rho]\!]_{\mathrm{inj}} \cap [\![\rho']\!]_{\mathrm{inj}} \neq \emptyset$ implies that $\rho = \rho'$ (actually $\rho \simeq \rho'$).

PROOF: The function $f : \rho \in \mathcal{T}(\pi) \mapsto [x]_{\sim_{\parr\Gamma}}$, where $x \in [\![\rho_{\mathrm{inj}}]\!]$ is bijective. Notice that by Fact 1 $[x]_{\sim_{\parr\Gamma}} = [\![\rho]\!]_{\mathrm{inj}} \subseteq [\![\pi]\!]_{\mathrm{inj}}$.

f injective: for $\rho \neq \rho'$ and $x \in [\![\rho]\!]_{\mathrm{inj}}$, $x' \in [\![\rho']\!]_{\mathrm{inj}}$, we have $x \not\sim_{\parr\Gamma} x'$, otherwise by Fact 1(ii) $x, x' \in [\![\rho]\!]_{\mathrm{inj}} \cap [\![\rho']\!]_{\mathrm{inj}}$ and by Fact 2 $\rho = \rho'$.
f surjective: for $[x]_{\sim_{\parr\Gamma}} \in [\![\pi]\!]_{\mathrm{inj}/\sim_{\parr\Gamma}}$ there is $\rho \in \mathcal{T}(\pi)$ s.t. $x \in [\![\rho]\!]_{\mathrm{inj}}$ and then $f(\rho) = [x]_{\sim_{\parr\Gamma}}(= [\![\rho]\!]_{\mathrm{inj}}$ by Fact 1).